Programming Languages and Compilers (CS 421)

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Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

Evaluating declarations

- Evaluation uses an environment ρ
- To evaluate a (simple) declaration let x = e
  - Evaluate expression e in ρ to value v
  - Then update ρ with x v: $\{x \to v\} + \rho$

Evaluating expressions

- Evaluation uses an environment ρ
- A constant evaluates to itself
- To evaluate an variable, look it up in ρ ($\rho(v)$)
- To evaluate uses of +, _ , etc, eval args, then do operation
- Function expression evaluates to its closure
- To evaluate a local dec: let x = e1 in e2
  - Eval e1 to v, then eval e2 using $\{x \rightarrow v\} + \rho$

Eval of App with Closures in OCaml

1. Evaluate the right term to values, $(v_1,\ldots,v_n)$
2. In environment ρ, evaluate left term to closure, $c = <(x_1,\ldots,x_n) \rightarrow b, \rho>$
3. Match $(x_1,\ldots,x_n)$ variables in (first) argument with values $(v_1,\ldots,v_n)$
4. Update the environment ρ to $\rho' = \{x_1 \rightarrow v_1,\ldots, x_n \rightarrow v_n\} + \rho$
5. Evaluate body b in environment ρ'

OCaml Example 1

```
# (print_string "a";
   (fun x -> (print_string "b";
              (fun y -> (print_string "c";
                         x + y)))))
  (print_string "d"; 3)
  (print_string "e"; 5);;
```

OCaml Example 1

```
# (print_string "a";
   (fun x -> (print_string "b";
                (fun y -> (print_string "c";
                           x + y))))
  (print_string "d"; 3)
  (print_string "e"; 5);;

edabc- : int = 8
#
```

Your turn now

Try Problem 1 on HW3

```
# let f =  (print_string "a";
            (fun x -> (print_string "b";
                       (fun y -> (print_string "c";
                                  x + y))))) in
let u =  (print_string "d"; 3) in
let g = f u in
let v = (print_string "e"; 5) in g v;;
```

```
# let f =  (print_string "a";
              (fun x -> (print_string "b";
                         (fun y -> (print_string "c";
                                    x + y))))) in
let u =  (print_string "d"; 3) in
let g = f u in
let v = (print_string "e"; 5) in g v;;
adbec- : int = 8
```

Higher Order Functions

- A function is *higher-order* if it takes a function as an argument or returns one as a result
- Example:

```
# let compose f g = fun x -> f (g x);;
val compose : ('a -> 'b) -> ('c -> 'a) -> 'c -> 'b = <fun>
```

- The type ('a -> 'b) -> ('c -> 'a) -> 'c -> 'b is a higher order type because of ('a -> 'b) and ('c -> 'a) and -> 'c -> 'b

Thrice

- Recall:

```
# let thrice f x = f (f (f x));;
val thrice : ('a -> 'a) -> 'a -> 'a = <fun>
```

- How do you write thrice with compose?

Thrice

- Recall:

```
# let thrice f x = f (f (f x));;
val thrice : ('a -> 'a) -> 'a -> 'a = <fun>
```

- How do you write thrice with compose?

```
# let thrice f = compose f (compose f f);;
val thrice : ('a -> 'a) -> 'a -> 'a = <fun>
```

- Is this the only way?

Partial Application

```
# (+);;
- : int -> int -> int = <fun>
# (+) 2 3;;
- : int = 5
# let plus_two = (+) 2;;
val plus_two : int -> int = <fun>
# plus_two 7;;
- : int = 9
```

- Patial application also called *sectioning*

Lambda Lifting

- You must remember the rules for evaluation when you use partial application

```
# let add_two = (+) (print_string "test\n"; 2);;
test
val add_two : int -> int = <fun>
# let add2 =      (* lambda lifted *)
   fun x -> (+) (print_string "test\n"; 2) x;;
val add2 : int -> int = <fun>
```

Lambda Lifting

```
# thrice add_two 5;;
- : int = 11
# thrice add2 5;;
test
test
test
- : int = 11
```

- Lambda lifting delayed the evaluation of the argument to (+) until the second argument was supplied

Partial Application and "Unknown Types"

- Recall compose plus_two:

```
# let f1 = compose plus_two;;
val f1 : ('_a -> int) -> '_a -> int = <fun>
```

- Compare to lambda lifted version:

```
# let f2 = fun g ->  compose plus_two g;;
val f2 : ('a -> int) -> 'a -> int = <fun>
```

- What is the difference?

Partial Application and “Unknown Types”

- ‘_a can only be instantiated once for an expression

```
# f1 plus_two;;
- : int -> int = <fun>
# f1 List.length;;
Characters 3-14:
  f1 List.length;;
     ^^^^^^^^^^^
This expression has type 'a list -> int but is here used
   with type int -> int
```

Partial Application and “Unknown Types”

- ‘a can be repeatedly instantiated

```
# f2 plus_two;;
- : int -> int = <fun>
# f2 List.length;;
- : '_a list -> int = <fun>
```

Your turn now

Try Problem 2 on HW3

Lists

- First example of a recursive datatype (aka algebraic datatype)
- Unlike tuples, lists are homogeneous in type (all elements same type)

Lists

- List can take one of two forms:
  - Empty list, written [ ]
  - Non-empty list, written x :: xs
    - x is head element, xs is tail list, :: called "cons"
  - Syntactic sugar: [x] == x :: [ ]
  - [x1; x2; …; xn] == x1 :: x2 :: … :: xn :: [ ]

Lists

```
# let fib5 = [8;5;3;2;1;1];;
val fib5 : int list = [8; 5; 3; 2; 1; 1]
# let fib6 = 13 :: fib5;;
val fib6 : int list = [13; 8; 5; 3; 2; 1; 1]
# (8::5::3::2::1::1::[ ]) = fib5;;
- : bool = true
# fib5 @ fib6;;
- : int list = [8; 5; 3; 2; 1; 1; 13; 8; 5; 3; 2; 1;
   1]
```

Lists are Homogeneous

```
# let bad_list = [1; 3.2; 7];;
Characters 19-22:
  let bad_list = [1; 3.2; 7];;
                     ^^^
This expression has type float but is here
  used with type int
```

Question

- Which one of these lists is invalid?

1. [2; 3; 4; 6]
2. [2,3; 4,5; 6,7]
3. [(2.3,4); (3.2,5); (6,7.2)]
4. [[“hi”; “there”]; [“wahcha”]; [ ]; [“doin”]]

Answer

- Which one of these lists is invalid?

1. [2; 3; 4; 6]
2. [2,3; 4,5; 6,7]
3. [(2.3,4); (3.2,5); (6,7.2)]
4. [[“hi”; “there”]; [“wahcha”]; [ ]; [“doin”]]

- 3 is invalid because of last pair

Functions Over Lists

```
# let rec double_up list =
    match list
    with [ ] -> [ ]  (* pattern before ->,
                        expression after *)
       | (x :: xs) -> (x :: x :: double_up xs);;
val double_up : 'a list -> 'a list = <fun>
# let fib5_2 = double_up fib5;;
val fib5_2 : int list = [8; 8; 5; 5; 3; 3; 2; 2; 1;
  1; 1; 1]
```

Functions Over Lists

```
# let silly = double_up ["hi"; "there"];;
val silly : string list = ["hi"; "hi"; "there"; "there"]
# let rec poor_rev list =
  match list
  with [] -> []
     | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
# poor_rev silly;;
- : string list = ["there"; "there"; "hi"; "hi"]
```

Question: Length of list

- Problem: write code for the length of the list
  - How to start?

```
let      length l =
```

Question: Length of list

- Problem: write code for the length of the list
  - How to start?

```
let rec length l =
   match l with
```

Question: Length of list

- Problem: write code for the length of the list
  - What patterns should we match against?

```
let rec length l =
    match l with
```

Question: Length of list

- Problem: write code for the length of the list
  - What patterns should we match against?

```
let rec length l =
    match l with [] ->
     | (a :: bs) ->
```

Question: Length of list

- Problem: write code for the length of the list
  - What result do we give when l is empty?

```
let rec length l =
    match l with [] ->
     | (a :: bs) ->
```

Question: Length of list

- Problem: write code for the length of the list
  - What result do we give when l is empty?

```
let rec length l =
    match l with [] -> 0
     | (a :: bs) ->
```

Question: Length of list

- Problem: write code for the length of the list
  - What result do we give when l is not empty?

```
let rec length l =
    match l with [] -> 0
     | (a :: bs) ->
```

Question: Length of list

- Problem: write code for the length of the list
  - What result do we give when l is not empty?

```
let rec length l =
    match l with [] -> 0
     | (a :: bs) -> 1 + length bs
```

Your turn now

Try Problem 1 on MP3

Same Length

- How can we efficiently answer if two lists have the same length?

Same Length

- How can we efficiently answer if two lists have the same length?

```
let rec same_length list1 list2 =
   match list1 with [] ->
      (match list2 with [] -> true
        | (y::ys) -> false)
   | (x::xs) ->
      (match list2 with [] -> false
        | (y::ys) -> same_length xs ys)
```

Structural Recursion

- Functions on recursive datatypes (eg lists) tend to be recursive
- Recursion over recursive datatypes generally by structural recursion
  - Recursive calls made to components of structure of the same recursive type
  - Base cases of recursive types stop the recursion of the function

Structural Recursion : List Example

```
# let rec length list = match list
  with [ ] -> 0   (* Nil case *)
  | x :: xs -> 1 + length xs;;  (* Cons case *)
val length : 'a list -> int = <fun>
# length [5; 4; 3; 2];;
- : int = 4
```

- Nil case [ ] is base case
- Cons case recurses on component list xs

Forward Recursion

- In Structural Recursion, split input into components and (eventually) recurse
- Forward Recursion form of Structural Recursion
- In forward recursion, first call the function recursively on all recursive components, and then build final result from partial results
- Wait until whole structure has been traversed to start building answer

Forward Recursion: Examples

```
# let rec double_up list =
    match list
    with [ ] -> [ ]
      | (x :: xs) -> (x :: x :: double_up xs);;
val double_up : 'a list -> 'a list = <fun>

# let rec poor_rev list =
  match list
  with [] -> []
    | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```

Forward Recursion: Example

```
# let rec map f list =
  match list
  with [] -> []
  | (h::t) -> (f h) :: (map f t);;
val map : ('a -> 'b) -> 'a list -> 'b list = <fun>
# map plus_two fib5;;
- : int list = [10; 7; 5; 4; 3; 3]
# map (fun x -> x - 1) fib6;;
: int list = [12; 7; 4; 2; 1; 0; 0]
```

Question

- How do you write length with forward recursion?

```
let rec length l =
```

Question

- How do you write length with forward recursion?

```
let rec length l =
    match l with [] ->
     | (a :: bs) ->
```

Question

- How do you write length with forward recursion?

```
let rec length l =
    match l with [] ->
     | (a :: bs) ->        length bs
```

Question

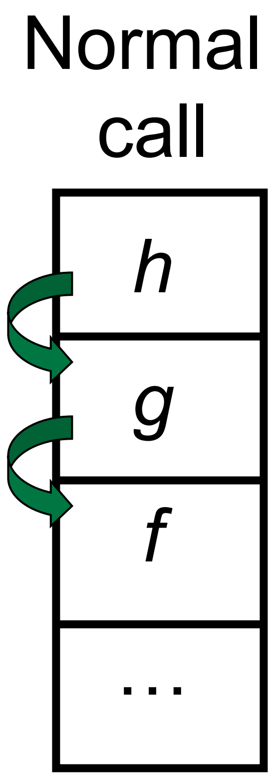
- How do you write length with forward recursion?

```
let rec length l =
    match l with [] -> 0
     | (a :: bs) -> 1 + length bs
```

Your turn now

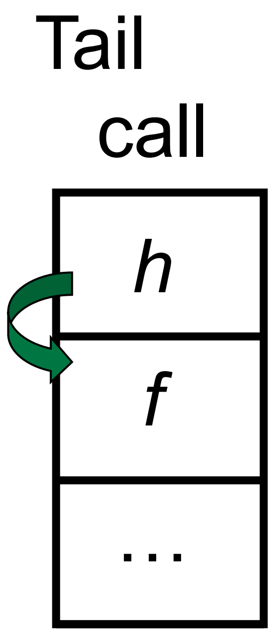
Try Problem 8 on MP3

An Important Optimization



- When a function call is made, the return address needs to be saved to the stack so we know to where to return when the call is finished
- What if *f* calls *g* and *g* calls *h*, but calling *h* is the last thing *g* does (a *tail call*)?

An Important Optimization



- When a function call is made, the return address needs to be saved to the stack so we know to where to return when the call is finished
- What if *f* calls *g* and *g* calls *h*, but calling *h* is the last thing *g* does (a *tail call*)?
- Then *h* can return directly to *f* instead of *g*

Tail Recursion

- A recursive program is tail recursive if all recursive calls are tail calls
- Tail recursive programs may be optimized to be implemented as loops, thus removing the function call overhead for the recursive calls
- Tail recursion generally requires extra "accumulator" arguments to pass partial results
  - May require an auxiliary function

Example of Tail Recursion

```
# let rec prod l =
    match l with [] -> 1
    | (x :: rem) -> x * prod rem;;
val prod : int list -> int = <fun>
# let prod list =
    let rec prod_aux l acc =
        match l with [] -> acc
        | (y :: rest) -> prod_aux rest (acc * y)
(* Uses associativity of multiplication *)
    in prod_aux list 1;;
 val prod : int list -> int = <fun>
```

Question

- How do you write length with tail recursion?

let length l =

Question

- How do you write length with tail recursion?

```
let length l =
    let rec length_aux list n =



in
```

Question

- How do you write length with tail recursion?

```
let length l =
    let rec length_aux list n =
    match list with [] ->
     | (a :: bs) ->
in
```

Question

- How do you write length with tail recursion?

```
let length l =
    let rec length_aux list n =
    match list with [] -> n
     | (a :: bs) ->
in
```

Question

- How do you write length with tail recursion?

```
let length l =
    let rec length_aux list n =
    match list with [] -> n
     | (a :: bs) -> length_aux
in
```

Question

- How do you write length with tail recursion?

```
let length l =
    let rec length_aux list n =
    match list with [] -> n
     | (a :: bs) -> length_aux bs
in
```

Question

- How do you write length with tail recursion?

```
let length l =
    let rec length_aux list n =
    match list with [] -> n
     | (a :: bs) -> length_aux bs (n + 1)
in
```

Question

- How do you write length with tail recursion?

```
let length l =
    let rec length_aux list n =
    match list with [] -> n
     | (a :: bs) -> length_aux bs (n + 1)
in length_aux l 0
```

Your turn now

Try Problem 10 on MP3

Mapping Functions Over Lists

```
# let rec map f list =
  match list
  with [] -> []
  | (h::t) -> (f h) :: (map f t);;
val map : ('a -> 'b) -> 'a list -> 'b list = <fun>
# map plus_two fib5;;
- : int list = [10; 7; 5; 4; 3; 3]
# map (fun x -> x - 1) fib6;;
: int list = [12; 7; 4; 2; 1; 0; 0]
```

Mapping Recursion

- One common form of structural recursion applies a function to each element in the structure

```
# let rec doubleList list = match list
   with [ ] -> [ ]
   | x::xs -> 2 * x :: doubleList xs;;
val doubleList : int list -> int list = <fun>
# doubleList [2;3;4];;
- : int list = [4; 6; 8]
```

Mapping Recursion

- Can use the higher-order recursive map function instead of direct recursion

```
# let doubleList list =
    List.map (fun x -> 2 * x) list;;
val doubleList : int list -> int list = <fun>
# doubleList [2;3;4];;
- : int list = [4; 6; 8]
```

- Same function, but no rec

Folding Recursion

- Another common form "folds" an operation over the elements of the structure

```
# let rec multList list = match list
  with [ ] -> 1
  | x::xs -> x * multList xs;;
val multList : int list -> int = <fun>
# multList [2;4;6];;
- : int = 48
```

- Computes (2 * (4 * (6 * 1)))

Folding Functions over Lists

How are the following functions similar?

```
# let rec sumlist list = match list with
  [ ] -> 0 | x::xs -> x + sumlist xs;;
val sumlist : int list -> int = <fun>
# sumlist [2;3;4];;
- : int = 9
# let rec prodlist list = match list with
  [ ] -> 1 | x::xs -> x * prodlist xs;;
val prodlist : int list -> int = <fun>
# prodlist [2;3;4];;
- : int = 24
```

Iterating over lists

```
# let rec fold_right f list b =
  match list
  with [] -> b
  | (x :: xs) -> f x (fold_right f xs b);;
val fold_right : ('a -> 'b -> 'b) -> 'a list -> 'b -> 'b =
  <fun>
# fold_right
    (fun s -> fun () -> print_string s)
    ["hi"; "there"]
    ();;
therehi- : unit = ()
```

Folding Recursion

- multList folds to the right
- Same as:

```
# let multList list =
    List.fold_right
    (fun x -> fun p -> x * p)
    list 1;;
val multList : int list -> int = <fun>
# multList [2;4;6];;
- : int = 48
```

Encoding Recursion with Fold

```
# let rec append list1 list2 = match list1 with
  [ ] -> list2 | x::xs -> x :: append xs list2;;
val append : 'a list -> 'a list -> 'a list = <fun>
```

Base Case | Operation | Recursive Call

```
# let append list1 list2 =
   fold_right (fun x y -> x :: y) list1 list2;;
val append : 'a list -> 'a list -> 'a list = <fun>
# append [1;2;3] [4;5;6];;
 - : int list = [1; 2; 3; 4; 5; 6]
```

Question

```
let rec length l =
    match l with [] -> 0
     | (a :: bs) -> 1 + length bs
```

- How do you write length with fold_right, but no explicit recursion?

Question

```
let rec length l =
    match l with [] -> 0
     | (a :: bs) -> 1 + length bs
```

- How do you write length with fold_right, but no explicit recursion?

```
let length list =
 List.fold_right (fun x -> fun n ->  n + 1) list 0
```

Map from Fold

```
# let map f list =
 fold_right (fun x -> fun y -> f x :: y) list
  [ ];;
val map : ('a -> 'b) -> 'a list -> 'b list =
  <fun>
# map ((+)1) [1;2;3];;
- : int list = [2; 3; 4]
```

- Can you write fold_right (or fold_left) with just map? How, or why not?

Iterating over lists

```
# let rec fold_left f a list =
  match list
  with [] -> a
  | (x :: xs) -> fold_left f (f a x) xs;;
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a =
   <fun>
# fold_left
   (fun () -> print_string)
   ()
   ["hi"; "there"];;
hithere- : unit = ()
```

Encoding Tail Recursion with fold_left

```
# let prod list = let rec prod_aux l acc =
      match l with [] -> acc
      | (y :: rest) -> prod_aux rest (acc * y)
    in prod_aux list 1;;
val prod : int list -> int = <fun>
```

Init Acc Value | Recursive Call | Operation

```
# let prod list =
  List.fold_left (fun acc y -> acc * y) 1 list;;
val prod: int list -> int = <fun>
# prod [4;5;6];;
 - : int =120
```

Question

```
let length l =
    let rec length_aux list n =
    match list with [] -> n
     | (a :: bs) -> length_aux bs (n + 1)
in length_aux l 0
```

- How do you write length with fold_left, but no explicit recursion?

Question

```
let length l =
    let rec length_aux list n =
    match list with [] -> n
     | (a :: bs) -> length_aux bs (n + 1)
in length_aux l 0
```

- How do you write length with fold_left, but no explicit recursion?

```
let length list =
 List.fold_left (fun n -> fun x -> n + 1) 0 list
```

Folding

```
# let rec fold_left f a list = match list
  with [] -> a | (x :: xs) -> fold_left f (f a x) xs;;
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a =
  <fun>
```

fold_left f a [x_1; x_2;…;x_n] = f(…(f (f a x_1) x_2)…)x_n

```
# let rec fold_right f list b = match list
  with [ ] -> b | (x :: xs) -> f x (fold_right f xs b);;
val fold_right : ('a -> 'b -> 'b) -> 'a list -> 'b -> 'b =
  <fun>
```

fold_right f [x_1; x_2;…;x_n] b = f x_1(f x_2 (…(f x_n b)…))

Recall

```
# let rec poor_rev list = match list
  with [] -> []
     | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```

- What is its running time?

Quadratic Time

- Each step of the recursion takes time proportional to input
- Each step of the recursion makes only one recursive call.
- List example:

```
# let rec poor_rev list = match list
  with [] -> []
    | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```

Tail Recursion - Example

```
# let rec rev_aux list revlist =
  match list with [ ] -> revlist
  | x :: xs -> rev_aux xs (x::revlist);;
val rev_aux : 'a list -> 'a list -> 'a list = <fun>

# let rev list = rev_aux list [ ];;
val rev : 'a list -> 'a list = <fun>
```

- What is its running time?

Comparison

- poor_rev [1,2,3] =
- (poor_rev [2,3]) @ [1] =
- ((poor_rev [3]) @ [2]) @ [1] =
- (((poor_rev [ ]) @ [3]) @ [2]) @ [1] =
- (([ ] @ [3]) @ [2]) @ [1]) =
- ([3] @ [2]) @ [1] =
- (3:: ([ ] @ [2])) @ [1] =
- [3,2] @ [1] =
- 3 :: ([2] @ [1]) =
- 3 :: (2:: ([ ] @ [1])) = [3, 2, 1]

Comparison

- rev [1,2,3] =
- rev_aux [1,2,3] [ ] =
- rev_aux [2,3] [1] =
- rev_aux [3] [2,1] =
- rev_aux [ ] [3,2,1] = [3,2,1]

Folding - Tail Recursion

```
# let rev list =
        fold_left
          (fun l -> fun x -> x :: l)     //comb op
         []                  //accumulator cell
         list
```

Folding

- Can replace recursion by fold_right in any forward primitive recursive definition
  - Primitive recursive means it only recurses on immediate subcomponents of recursive data structure
- Can replace recursion by fold_left in any tail primitive recursive definition