Programming Languages and Compilers (CS 421)

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Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

Curry - Howard Isomorphism

- Type Systems are logics; logics are type systems
- Types are propositions; propositions are types
- Terms are proofs; proofs are terms

- Functions space arrow corresponds to implication; application corresponds to modus ponens

Curry - Howard Isomorphism

- Modus Ponens

$$\frac{A \Rightarrow B \quad A}{B}$$

- Application

$$\frac{\Gamma \mid\!\text{-}\; e_1 : \alpha \to \beta \quad \Gamma \mid\!\text{-}\; e_2 : \alpha}{\Gamma \mid\!\text{-}\; (e_1\; e_2) : \beta}$$

Mia Copa

- The above system can't handle polymorphism as in OCAML
- No type variables in type language (only meta-variable in the logic)
- Would need:
  - Object level type variables and some kind of type quantification
  - **let** and **let rec** rules to introduce polymorphism
  - Explicit rule to eliminate (instantiate) polymorphism

Support for Polymorphic Types

- Monomorpic Types (τ):
  - Basic Types: int, bool, float, string, unit, …
  - Type Variables: α, β, γ δ ε
  - Compound Types: $\alpha \rightarrow \beta$, int * string, bool list, …
- Polymorphic Types:
  - Monomorphic types τ
  - Universally quantified monomorphic types
  - $\forall \alpha_1, \ldots, \alpha_n . \tau$
  - Can think of τ as same as $\forall . \tau$

Support for Polymorphic Types

- Typing Environment Γ supplies polymorphic types (which will often just be monomorphic) for variables
- Free variables of monomorphic type just type variables that occur in it
  - Write FreeVars(τ)
- Free variables of polymorphic type removes variables that are universally quantified
  - FreeVars($\forall\alpha_1, \ldots, \alpha_n . \tau$) = FreeVars($\tau$) – $\{\alpha_1, \ldots, \alpha_n\}$
- FreeVars(Γ) = all FreeVars of types in range of Γ

Monomorphic to Polymorphic

- Given:
  - type environment Γ
  - monomorphic type τ
  - τ shares type variables with Γ
- Want most polymorphic type for τ that doesn't break sharing type variables with Γ
- $\mathrm{Gen}(\tau, \Gamma) = \forall\, \alpha_1, \ldots, \alpha_n \,.\, \tau$ where $\{\alpha_1, \ldots, \alpha_n\} = \mathrm{freeVars}(\tau) - \mathrm{freeVars}(\Gamma)$

Polymorphic Typing Rules

- A *type judgement* has the form

$$\Gamma \text{ |- exp} : \tau$$

  - Γ uses polymorphic types
  - τ still monomorphic
- Most rules stay same (except use more general typing environments)
- Rules that change:
  - Variables
  - Let and Let Rec
  - Allow polymorphic constants
- Worth noting functions again

Polymorphic Let and Let Rec

- let rule:

$$\frac{\Gamma \mid\!- \; e_1 : \tau_1 \quad [x : \mathrm{Gen}(\tau_1,\Gamma)] + \Gamma \mid\!- \; e_2 : \tau_2}{\Gamma \mid\!- \; (\text{let } x = e_1 \text{ in } e_2) : \tau_2}$$

- let rec rule:

$$\frac{[x : \tau_1] + \Gamma \mid\!- \; e_1 : \tau_1 \quad [x : \mathrm{Gen}(\tau_1,\Gamma)] + \Gamma \mid\!- \; e_2 : \tau_2}{\Gamma \mid\!- \; (\text{let rec } x = e_1 \text{ in } e_2) : \tau_2}$$

Polymorphic Variables (Identifiers)

Variable axiom:

$$\frac{}{\Gamma \mid\!- \ x : \varphi(\tau)} \quad \text{if } \Gamma(x) = \forall \alpha_1, \ldots, \alpha_n . \tau$$

- Where φ replaces all occurrences of $\alpha_1, \ldots, \alpha_n$ by monotypes $\tau_1, \ldots, \tau_n$
- Note: Monomorphic rule special case:

$$\frac{}{\Gamma \mid\!- \ x : \tau} \quad \text{if } \Gamma(x) = \tau$$

- Constants treated same way

Fun Rule Stays the Same

- fun rule:

$$\frac{[x : \tau_1] + \Gamma \mid\!\!- \; e : \tau_2}{\Gamma \mid\!\!- \text{fun } x \text{ -> } e : \tau_1 \rightarrow \tau_2}$$

- Types τ_1, τ_2 monomorphic
- Function argument must always be used at same type in function body

Polymorphic Example

- Assume additional constants:
- hd :$\forall\alpha.\ \alpha$ list -> α
- tl: $\forall\alpha.\ \alpha$ list -> α list
- is_empty : $\forall\alpha.\ \alpha$ list -> bool
- :: : $\forall\alpha.\ \alpha$ -> α list -> α list
- [] : $\forall\alpha.\ \alpha$ list

Polymorphic Example

- Show:

?

{} |- let rec length =

fun l -> if is_empty l then 0

else 1 + length (tl l)

in length ((::) 2 []) + length((::) true []) : int

Polymorphic Example: Let Rec Rule

- Show: (1) (2)

$$\dfrac{\{length:\alpha \text{ list} \to \text{int}\} \mid\!- \text{fun } l \to \ldots : \alpha \text{ list} \to \text{int} \qquad \{length:\forall\alpha.\ \alpha \text{ list} \to \text{int}\} \mid\!- \text{length } ((::)\ 2\ [\,]) + \text{length}((::)\ \text{true}\ [\,]) : \text{int}}{\{\} \mid\!- \text{let rec length} = \text{fun } l \to \text{if is\_empty } l \text{ then } 0 \text{ else } 1 + \text{length (tl } l) \text{ in length } ((::)\ 2\ [\,]) + \text{length}((::)\ \text{true}\ [\,]) : \text{int}}$$

Polymorphic Example (1)

- Show:

$$\frac{?}{\{\text{length}:\alpha\ \text{list} \to \text{int}\} \vdash}$$

{length:α list -> int} |-

fun l -> if is_empty l then 0

else 1 + length (tl l)

: α list -> int

Polymorphic Example (1): Fun Rule

- Show: (3)

$\{$length:α list -> int, l: α list $\}$ |-

if is_empty l then 0

else length (hd l) + length (tl l) : int

$\{$length:α list -> int$\}$ |-

fun l -> if is_empty l then 0

else 1 + length (tl l)

: α list -> int

Polymorphic Example (3)

- Let Γ ={length:α list -> int, l: α list }
- Show

?

$$\frac{?}{\Gamma \mid\!- \text{ if is\_empty l then 0 else 1 + length (tl l) : int}}$$

Polymorphic Example (3):IfThenElse

- Let Γ ={length:α list -> int, l: α list }
- Show

$$\frac{\begin{array}{ccc} (4) & (5) & (6) \\ \Gamma \vdash \text{is\_empty l : bool} & \Gamma \vdash 0\text{:int} & \Gamma \vdash 1 + \text{length (tl l) : int} \end{array}}{\Gamma \vdash \text{if is\_empty l then 0 else 1 + length (tl l) : int}}$$

Polymorphic Example (4)

- Let Γ ={length:α list -> int, l: α list }
- Show

$$\frac{?}{\Gamma \text{|- is\_empty l : bool}}$$

Polymorphic Example (4):Application

- Let Γ ={length:α list -> int, l: α list }
- Show

$$\dfrac{\dfrac{?}{\Gamma |\text{- is\_empty} : \alpha \text{ list -> bool}} \qquad \dfrac{?}{\Gamma |\text{- l} : \alpha \text{ list}}}{\Gamma |\text{- is\_empty l} : \text{bool}}$$

Polymorphic Example (4)

- Let Γ ={length:α list -> int, l: α list }
- Show

By Const since α list -> bool is instance of $\forall\alpha.\ \alpha$ list -> bool ?

$$\dfrac{\dfrac{}{\Gamma \mid\!- \text{ is\_empty} : \alpha \text{ list} \to \text{bool}} \qquad \dfrac{}{\Gamma \mid\!- \text{ l} : \alpha \text{ list}}}{\Gamma \mid\!- \text{ is\_empty l} : \text{bool}}$$

Polymorphic Example (4)

- Let Γ ={length:α list -> int, l: α list }
- Show

$$\frac{\dfrac{\text{By Const since } \alpha \text{ list -> bool is instance of } \forall\alpha.\ \alpha \text{ list -> bool}}{\Gamma \text{|- is\_empty : } \alpha \text{ list -> bool}} \qquad \dfrac{\text{By Variable } \Gamma(\text{l}) = \alpha \text{ list}}{\Gamma \text{|- l : } \alpha \text{ list}}}{\Gamma \text{|- is\_empty l : bool}}$$

- This finishes (4)

Polymorphic Example (5):Const

- Let Γ ={length:α list -> int, l: α list }
- Show

By Const Rule

$$\frac{}{\Gamma |\text{-}\ 0:\text{int}}$$

Polymorphic Example (6):Arith Op

- Let Γ ={length:α list -> int, l: α list }
- Show

$$
\dfrac{\dfrac{\text{By Const}}{\Gamma |\!-\ 1:\text{int}} \qquad \dfrac{\dfrac{\text{By Variable}}{\Gamma |\!-\ \text{length} : \alpha\ \text{list} \to \text{int}} \quad \text{(7)} \quad \Gamma |\!-\ (\text{tl l}) : \alpha\ \text{list}}{\Gamma |\!-\ \text{length (tl l)} : \text{int}}}{\Gamma |\!-\ 1 + \text{length (tl l)} : \text{int}}
$$

Polymorphic Example (7):App Rule

- Let Γ ={length:α list -> int, l: α list }
- Show

$$\frac{\dfrac{\text{By Const}}{\Gamma|\text{- (tl l) : } \alpha \text{ list -> } \alpha \text{ list}} \qquad \dfrac{\text{By Variable}}{\Gamma|\text{- l : } \alpha \text{ list}}}{\Gamma|\text{- (tl l) : } \alpha \text{ list}}$$

By Const since α list -> α list is instance of $\forall\alpha.$ α list -> α list

Polymorphic Example: (2) by ArithOp

- Let Γ′ = {length:∀α. α list -> int}
- Show:

$$\frac{\begin{array}{cc} (8) & (9) \\ \Gamma' \text{ |- } & \Gamma' \text{ |- } \\ \text{length ((::) 2 []) :int} & \text{length((::) true []) : int} \end{array}}{\begin{array}{l} \text{\{length: } \alpha.\ \alpha \text{ list -> int\}} \\ \text{|- length ((::) 2 []) + length((::) true []) : int} \end{array}}$$

Polymorphic Example: (8)AppRule

- Let Γ' = {length:$\forall\alpha.\ \alpha$ list -> int}
- Show:

$$\frac{\Gamma' \text{ |- length : int list ->int} \quad \Gamma' \text{ |- ((::)2 []):int list}}{\Gamma' \text{ |- length ((::) 2 []) :int}}$$

Polymorphic Example: (8)AppRule

- Let Γ' = {length:$\forall\alpha.\ \alpha$ list -> int}
- Show:

By Var since int list -> int is instance of

$\forall\alpha.\ \alpha$ list -> int

$$\dfrac{\dfrac{}{\Gamma' \text{ |- length : int list ->int}} \qquad \dfrac{(10)}{\Gamma' \text{ |- ((::)2 []):int list}}}{\Gamma' \text{ |- length ((::) 2 []) :int}}$$

Polymorphic Example: (10)AppRule

- Let Γ' = {length:$\forall\alpha.\ \alpha$ list -> int}
- Show:
- By Const since α list is instance of $\forall\alpha.\ \alpha$ list

$$\dfrac{\quad(11)\quad \Gamma'\text{|-((::) 2) : int list -> int list} \qquad \dfrac{}{\Gamma'\text{ |- [] : int list}}}{\Gamma'\text{ |- ((::) 2 []) :int list}}$$

Polymorphic Example: (11)AppRule

- Let $\Gamma' = \{\text{length}:\forall\alpha.\ \alpha \text{ list} \to \text{int}\}$
- Show:
- By Const since α list is instance of $\forall\alpha.\ \alpha$ list

$$\dfrac{\dfrac{\forall\alpha.\ \alpha \text{ list}}{\Gamma' \mid\!- (::) : \text{int} \to \text{int list} \to \text{int list}} \qquad \dfrac{\text{By Const}}{\Gamma' \mid\!- 2 : \text{int}}}{\Gamma' \mid\!- ((::)\ 2) : \text{int list} \to \text{int list}}$$

Polymorphic Example: (9)AppRule

- Let Γ' = {length:$\forall\alpha.\ \alpha$ list -> int}
- Show:

$$\frac{\Gamma' \text{ |- } \text{length:bool list ->int} \qquad \Gamma' \text{ |- } \text{((::) true []):bool list}}{\Gamma' \text{ |- length ((::) true []) :int}}$$

Polymorphic Example: (9)AppRule

- Let Γ' = {length:$\forall\alpha.\ \alpha$ list -> int}
- Show:

By Var since bool list -> int is instance of

$\forall\alpha.\ \alpha$ list -> int

$$\dfrac{\dfrac{}{\Gamma' \text{ |- length:bool list ->int}} \qquad \dfrac{(12)}{\Gamma' \text{ |- ((::) true []):bool list}}}{\Gamma' \text{ |- length ((::) true []) :int}}$$

Polymorphic Example: (12)AppRule

- Let Γ' = {length:$\forall\alpha.\ \alpha$ list -> int}
- Show:
- By Const since α list is instance of $\forall\alpha.\ \alpha$ list

(13)

$$\frac{\Gamma' \mid\!- ((::)\text{true}):\text{bool list} \to \text{bool list} \quad \dfrac{}{\Gamma' \mid\!- [\,]:\text{bool list}}}{\Gamma' \mid\!- ((::)\ \text{true}\ [\,]) : \text{bool list}}$$

Polymorphic Example: (13)AppRule

- Let Γ' = {length:$\forall\alpha.\ \alpha$ list -> int}
- Show:

By Const since bool list

is instance of $\forall\alpha.\ \alpha$ list — By Const

$$\dfrac{\dfrac{\text{By Const since bool list is instance of } \forall\alpha.\ \alpha \text{ list}}{\Gamma' \text{ |- } (::):\text{bool ->bool list ->bool list}} \quad \dfrac{\text{By Const}}{\Gamma' \text{ |- true : bool}}}{\Gamma' \text{ |- ((::) true) : bool list -> bool list}}$$