

Prove that each of the following languages is *not* regular.

1.  $\{\emptyset^{2^n} \mid n \geq 0\}$

**Solution (fooling set  $F = L$ ):** Let  $F = L = \{\emptyset^{2^n} \mid n \geq 0\}$ .

Let  $x$  and  $y$  be arbitrary distinct elements of  $F$ .

Then  $x = \emptyset^{2^i}$  and  $y = \emptyset^{2^j}$  for some non-negative integers  $i \neq j$ .

Without loss of generality, assume  $i < j$ . (Otherwise, swap  $x$  and  $y$ .)

Let  $z = \emptyset^{2^i}$ .

- $xz = \emptyset^{2^i} \emptyset^{2^i} = \emptyset^{2^{i+1}} \in L$ .
- $yz = \emptyset^{2^j} \emptyset^{2^i} = \emptyset^{2^i+2^j}$ . The integer  $2^i + 2^j$  lies strictly between  $2^j$  and  $2^{j+1}$  (because  $i < j$ ) and thus is not a power of 2. It follows that  $yz \notin L$ .

Because  $xz \in L$  and  $yz \notin L$ , the suffix  $z$  distinguishes  $x$  and  $y$ .

We conclude that  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

**Solution (fooling set  $F = \emptyset^*$ ):** Let  $F = \emptyset^* = \{\emptyset^n \mid n \geq 0\}$ .

Let  $x$  and  $y$  be arbitrary distinct elements of  $F$ .

Then  $x = \emptyset^i$  and  $y = \emptyset^j$  for some non-negative integers  $i \neq j$ .

Without loss of generality, assume  $i < j$ . (Otherwise, swap  $x$  and  $y$ .)

Let  $r$  be any integer such that  $2^r > j$ , and let  $z = \emptyset^{2^r-i}$ .

Then  $xz = \emptyset^i \emptyset^{2^r-i} = \emptyset^{2^r} \in L$ .

But  $yz = \emptyset^j \emptyset^{2^r-i} = \emptyset^{2^r+j-i} \notin L$ , because  $2^r + j - i$  is not a power of 2:

$$\begin{aligned} 2^r &< 2^r + j - i && [i < j] \\ &\leq 2^r + j && [i \geq 0] \\ &< 2^r + 2^r && [j < 2^r] \\ &= 2^{r+1} && [\text{math}] \end{aligned}$$

We conclude that  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

**Solution (fooling set  $F = \emptyset^*$ , but distinguish the other way):**

Let  $F = \emptyset^* = \{\emptyset^n \mid n \geq 0\}$ .

Let  $x$  and  $y$  be arbitrary distinct elements of  $F$ .

Then  $x = \emptyset^i$  and  $y = \emptyset^j$  for some non-negative integers  $i \neq j$ .

Without loss of generality, assume  $i < j$ . (Otherwise, swap  $x$  and  $y$ .)

Let  $r$  be any integer such that  $2^{r-1} > j$ , and let  $z = \emptyset^{2^r-j}$ .

Then  $xz = 0^i 0^{2^r-j} = 0^{2^r-j+i} \notin L$ , because  $2^r - j + i$  is not a power of 2:

$$\begin{aligned} 2^{r-1} &= 2^r - 2^{r-1} && \text{[math]} \\ &< 2^r - j && \text{[} 2^{r-1} > j \text{]} \\ &\leq 2^r - j + i && \text{[} i \geq 0 \text{]} \\ &< 2^r && \text{[} i < j \text{]} \end{aligned}$$

But  $yz = 0^j 0^{2^r-j} = 0^{2^r} \in L$ .

We conclude that  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

2.  $\{0^{2n}1^n \mid n \geq 0\}$

**Solution (fooling set  $F = (00)^*$ ):** Let  $F$  be the language  $(00)^*$ .

Let  $x$  and  $y$  be arbitrary distinct strings in  $F$ .

Then  $x = 0^{2i}$  and  $y = 0^{2j}$  for some non-negative integers  $i \neq j$ .

Let  $z = 1^i$ .

Then  $xz = 0^{2i}1^i \in L$ .

And  $yz = 0^{2j}1^i \notin L$ , because  $i \neq j$ .

We conclude that  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

**Solution (fooling set  $F = 0^*$ ):** Let  $F$  be the language  $0^*$ .

Let  $x$  and  $y$  be arbitrary distinct strings in  $F$ .

Then  $x = 0^i$  and  $y = 0^j$  for some non-negative integers  $i \neq j$ .

Let  $z = 0^i1^i$ .

Then  $xz = 0^{2i}1^i \in L$ .

And  $yz = 0^{i+j}1^i \notin L$ , because  $i + j \neq 2i$ .

We conclude that  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

3.  $\{\emptyset^m 1^n \mid m \neq 2n\}$

**Solution (fooling set  $F = (\emptyset\emptyset)^*$ ):** Let  $F$  be the language  $(\emptyset\emptyset)^*$ .

Let  $x$  and  $y$  be arbitrary distinct strings in  $F$ .

Then  $x = \emptyset^{2i}$  and  $y = \emptyset^{2j}$  for some non-negative integers  $i \neq j$ .

Let  $z = 1^i$ .

Then  $xz = \emptyset^{2i} 1^i \notin L$ .

And  $yz = \emptyset^{2j} 1^i \in L$ , because  $i \neq j$ .

We conclude that  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

**Solution (fooling set  $F = \emptyset^*$ ):** Let  $F$  be the language  $\emptyset^*$ .

Let  $x$  and  $y$  be arbitrary distinct strings in  $F$ .

Then  $x = \emptyset^i$  and  $y = \emptyset^j$  for some non-negative integers  $i \neq j$ .

Let  $z = \emptyset^i 1^i$ .

Then  $xz = \emptyset^{2i} 1^i \notin L$ .

And  $yz = \emptyset^{i+j} 1^i \in L$ , because  $i + j \neq 2i$ .

We conclude that  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

**Solution (reduction via closure):** If  $L$  were regular, then the language

$$\emptyset^* 1^* \setminus L = \{\emptyset^m 1^n \mid m = 2n\} = \{\emptyset^{2n} 1^n \mid n \geq 0\}$$

would also be regular, because regular languages are closed under complement. But we just proved that  $\{\emptyset^{2n} 1^n \mid n \geq 0\}$  is not regular in problem 2. ■

4. Strings over  $\{0, 1\}$  where the number of 0s is exactly twice the number of 1s.

**Solution (fooling set  $F = 1^*$ ):** Let  $F$  be the language  $1^*$ .

Let  $x$  and  $y$  be arbitrary distinct strings in  $F$ .

Then  $x = 1^i$  and  $y = 1^j$  for some non-negative integers  $i \neq j$ .

Let  $z = 0^{2i}$ .

Then  $xz = 1^i 0^{2i} \in L$ .

And  $yz = 1^j 0^{2i} \notin L$ , because  $i \neq j$ .

We conclude that  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

**Solution (fooling set  $F = 0^*$ ):** Let  $F$  be the language  $0^*$ .

Let  $x$  and  $y$  be arbitrary distinct strings in  $F$ .

Then  $x = 0^i$  and  $y = 0^j$  for some non-negative integers  $i \neq j$ .

Let  $z = 0^i 1^i$ .

Then  $xz = 0^{2i} 1^i \in L$ .

And  $yz = 0^{i+j} 1^i \notin L$ , because  $i + j \neq 2i$ .

We conclude that  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

**Solution (reduction via closure):** If  $L$  were regular, then the language

$$L \cap 0^* 1^* = \{0^{2n} 1^n \mid n \geq 0\}$$

would also be regular, because regular languages are closed under intersection. But we just proved that  $\{0^{2n} 1^n \mid n \geq 0\}$  is not regular in problem 2. ■

5. Strings of properly nested parentheses  $()$ , brackets  $[\ ]$ , and braces  $\{\}$ . For example, the string  $([\ ])\{\}$  is in this language, but the string  $([\ ])$  is not, because the left and right delimiters don't match.

**Solution (fooling set):**

Let  $F$  be the language  $(^*$ .

Let  $x$  and  $y$  be arbitrary distinct strings in  $F$ .

Then  $x = (^i$  and  $y = (^j$  for some non-negative integers  $i \neq j$ .

Let  $z = )^i$ .

Then  $xz = (^i)^i \in L$ .

And  $yz = (^j)^i \notin L$ , because  $i \neq j$ .

We conclude that  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

*[Notice that this argument doesn't even try to consider strings with different types of brackets, because it doesn't have to. We proved that  $L$  has an infinite fooling set  $F$ ; that's enough.]*

**Solution (reduction via closure and renaming):** If  $L$  were regular, then the language

$$L' := L \cap [^*]^* = \{[{}^n]{}^n \mid n \geq 0\}$$

would also be regular, because regular languages are closed under intersection. But  $L'$  is the same as the language  $\{0^n 1^n \mid n \geq 0\}$ , except for renaming the symbols  $0 \mapsto [$  and  $1 \mapsto ]$ , and we proved that  $\{0^n 1^n \mid n \geq 0\}$  is not regular. ■

## Harder problems to think about later:

6. Strings of the form  $w_1\#w_2\#\dots\#w_n$  for some  $n \geq 2$ , where each substring  $w_i$  is a string in  $\{0, 1\}^*$ , and some pair of substrings  $w_i$  and  $w_j$  are equal.

**Solution (fooling set for the special case  $n = 2$ ):**

Let  $F$  be the language  $0^*$ .

Let  $x$  and  $y$  be arbitrary distinct strings in  $F$ .

Then  $x = 0^i$  and  $y = 0^j$  for some non-negative integers  $i \neq j$ .

Let  $z = \#0^i$ .

Then  $xz = 0^i\#0^i \in L$ .

And  $yz = 0^j\#0^i \notin L$ , because  $i \neq j$ .

We conclude that  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

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*Notice that this argument doesn't even try to consider strings with more than one #, because it doesn't have to. We proved that  $L$  has an infinite fooling set  $F$ ; that's enough.*

7.  $\{w \in (0 + 1)^* \mid w = x1^n \text{ for some string } x \text{ with } |x| = n\}$ , or less formally, binary strings whose right half contains only 1s.

**Solution (Fooling set  $F = 0^*$ ):** Let  $F$  be the language  $0^*$ .

Let  $x$  and  $y$  be arbitrary distinct strings in  $F$ .

Then  $x = 0^i$  and  $y = 0^j$  for some non-negative integers  $i \neq j$ .

Without loss of generality, assume  $i < j$  (otherwise swap).

Let  $z = 1^i$ .

Then  $xz = 0^i1^i \in L$ .

Also  $yz = 0^j1^i \notin L$ . There are two cases to consider:

- If  $|yz| = i + j$  is odd, then  $yz$  is not in  $L$ .
- Suppose  $|yz| = i + j$  is even. The right half of  $yz$  has length  $(i + j)/2 > i$  and thus contains at least one 0, so again  $yz$  is not in  $L$ .

We conclude that  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■

8.  $\{\emptyset^{n^2} \mid n \geq 0\}$

**Solution (fooling set  $F = L$ ):** Let  $x$  and  $y$  be arbitrary distinct strings in  $L$ .

Without loss of generality,  $x = \emptyset^{i^2}$  and  $y = \emptyset^{j^2}$  for some  $i > j \geq 0$ .

Let  $z = \emptyset^{2j+1}$ .

Then  $xz = \emptyset^{i^2+2j+1} \notin L$ , because  $i^2 < i^2 + 2j + 1 < (i + 1)^2$ .

On the other hand,  $yz = \emptyset^{j^2+2j+1} = \emptyset^{(j+1)^2} \in L$ .

Thus,  $z$  distinguishes  $x$  and  $y$ .

We conclude that  $L$  is a fooling set for  $L$ .

Because  $L$  is infinite,  $L$  cannot be regular. ■

**Solution (fooling set  $F = \emptyset^*$ ):** Let  $x$  and  $y$  be arbitrary distinct strings in  $\emptyset^*$ .

Without loss of generality,  $x = \emptyset^i$  and  $y = \emptyset^j$  for some  $i > j \geq 0$ .

Let  $z = \emptyset^{i^2+i+1}$ .

Then  $xz = \emptyset^{i^2+2i+1} = \emptyset^{(i+1)^2} \in L$ .

On the other hand,  $yz = \emptyset^{i^2+i+j+1} \notin L$ , because  $i^2 < i^2 + i + j + 1 < (i + 1)^2$ .

Thus,  $z$  distinguishes  $x$  and  $y$ .

We conclude that  $\emptyset^*$  is a fooling set for  $L$ .

Because  $\emptyset^*$  is infinite,  $L$  cannot be regular. ■

**Solution (fooling set  $F = \emptyset\emptyset\emptyset^*$ ):** Let  $x$  and  $y$  be arbitrary distinct strings in  $\emptyset\emptyset\emptyset^*$ .

Without loss of generality,  $x = \emptyset^i$  and  $y = \emptyset^j$  for some  $i > j \geq 3$ .

Let  $z = \emptyset^{i^2-i}$ .

Then  $xz = \emptyset^{i^2} \in L$ .

On the other hand,  $yz = \emptyset^{i^2-i+j} \notin L$ , because

$$(i-1)^2 = i^2 - 2i + 1 < i^2 - i < i^2 - i + j < i^2.$$

(The first inequality requires  $i \geq 2$ , and the second requires  $j \geq 1$ .)

Thus,  $z$  distinguishes  $x$  and  $y$ .

We conclude that  $\emptyset\emptyset\emptyset^*$  is a fooling set for  $L$ .

Because  $\emptyset\emptyset\emptyset^*$  is infinite,  $L$  cannot be regular. ■

\*9.  $\{w \in (\emptyset + 1)^* \mid w \text{ is the binary representation of a perfect square}\}$

**Solution (fooling set):** We design an infinite fooling set around numbers of the form  $(2^k + 1)^2 = 2^{2k} + 2^{k+1} + 1 = 1\emptyset^{k-2}1\emptyset^k1 \in L$ , for any integer  $k \geq 2$ . The argument is somewhat simpler if we further restrict  $k$  to be even.

Let  $F = 1(\emptyset\emptyset)^*1$ , and let  $x$  and  $y$  be arbitrary distinct strings in  $F$ .

Then  $x = 1\emptyset^{2i-2}1$  and  $y = 1\emptyset^{2j-2}1$ , for some positive integers  $i \neq j$ .

Without loss of generality, assume  $i < j$ . (Otherwise, swap  $x$  and  $y$ .)

Let  $z = \emptyset^{2i}1$ .

Then  $xz = 1\emptyset^{2i-2}1\emptyset^{2i}1$  is the binary representation of  $2^{4i} + 2^{2i+1} + 1 = (2^{2i} + 1)^2$ , and therefore  $xz \in L$ .

On the other hand,  $yz = 1\emptyset^{2j-2}1\emptyset^{2i}1$  is the binary representation of the integer  $2^{2i+2j} + 2^{2i+1} + 1$ . Simple algebra gives us the inequalities

$$\begin{aligned} (2^{i+j})^2 &= 2^{2i+2j} \\ &< 2^{2i+2j} + 2^{2i+1} + 1 \\ &< 2^{2(i+j)} + 2^{i+j+1} + 1 \\ &= (2^{i+j} + 1)^2. \end{aligned}$$

So  $2^{2i+2j} + 2^{2i+1} + 1$  lies between two consecutive perfect squares, and thus is not a perfect square, which implies that  $yz \notin L$ .

We conclude that  $F$  is a fooling set for  $L$ .

Because  $F$  is infinite,  $L$  cannot be regular. ■