Here are the formal recursive definitions of string length, concatenation, and reversal:

$$|w| := \begin{cases} 0 & \text{if } w = \varepsilon \\ 1 + |x| & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

$$w \bullet z := \begin{cases} z & \text{if } w = \varepsilon \\ a \cdot (x \bullet z) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

$$w^R := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ x^R \bullet a & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

Lemma 1: $w \bullet \varepsilon = w$ for all strings w.

Lemma 2: $|w \bullet z| = |w| + |z|$ for all strings w and z.

Lemma 3: $(w \bullet y) \bullet z = w \bullet (y \bullet z)$ for all strings w, y, and z.

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1. Prove that $|w^R| = |w|$ for every string w.

Solution (induction on w):

Let w be an arbitrary string.

Assume for every string x where $|x| < |w|$ that $|x^R| = |x|$.

There are two cases to consider.

- If $w = \varepsilon$, then

$$\begin{aligned} |w^R| &= |\varepsilon^R| && \text{because } w = \varepsilon \\ &= |\varepsilon| && \text{by definition of }^R \\ &= |w| && \text{because } w = \varepsilon \end{aligned}$$

- Otherwise, $w = ax$ for some symbol a and some string x, and therefore

$$\begin{aligned} |w^R| &= |(ax)^R| && \text{because } w = ax \\ &= |x^R \bullet a| && \text{by definition of }^R \\ &= |x^R| + |a| && \text{by Lemma 2} \\ &= |x^R| + 1 + |\varepsilon| && \text{by definition of } |\cdot| \\ &= |x^R| + 1 && \text{by definition of } |\cdot| \\ &= |x| + 1 && \text{by the induction hypothesis} \\ &= |ax| && \text{by definition of } |\cdot| \\ &= |w| && \text{because } w = ax \end{aligned}$$

In both cases, we conclude that $|w^R| = |w|$. ■

2. Prove that $(w \bullet z)^R = z^R \bullet w^R$ for all strings w and z.

Solution (induction on w):

Let w and z be arbitrary strings.

Assume for every string x where $|x| < |w|$ that $(x \bullet z)^R = x^R \bullet z^R$.

There are two cases to consider:

- If $w = \varepsilon$, then

$$\begin{aligned}
(w \bullet z)^R &= (\varepsilon \bullet z)^R && \text{because } w = \varepsilon \\
&= z^R && \text{by definition of } \bullet \\
&= z^R \bullet \varepsilon && \text{by Lemma 1} \\
&= z^R \bullet \varepsilon^R && \text{by definition of }^R \\
&= z^R \bullet w^R && \text{because } w = \varepsilon
\end{aligned}$$

- Otherwise, $w = ax$ for some symbol a and some string x.

$$\begin{aligned}
(w \bullet z)^R &= (ax \bullet z)^R && \text{because } w = ax \\
&= (a \cdot (x \bullet z))^R && \text{by definition of } \bullet \\
&= (x \bullet z)^R \bullet a && \text{by definition of }^R \\
&= (z^R \bullet x^R) \bullet a && \text{by the induction hypothesis } (\star) \\
&= z^R \bullet (x^R \bullet a) && \text{by Lemma 3} \\
&= z^R \bullet (ax)^R && \text{by definition of }^R \\
&= z^R \bullet w^R && \text{because } w = ax
\end{aligned}$$

In both cases, we conclude that $(w \bullet z)^R = z^R \bullet w^R$. ■

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*How did I know that the induction hypothesis needs to change the first string w, but not the second string z? I actually wrote down the inductive **argument** first, and then **noticed** that I needed to argue inductively about $x \bullet z$ at line $(\star)$. Same string z, but w changed to x.*

Alternatively, we could notice that the recursive definition of $w \bullet z$ recurses on w but leaves z unchanged. Inductive proofs always mirror the recursive definitions of the objects in question.

*Alternatively, in light of Lemma 2, we could have used induction on the **sum** of the string lengths. Then the inductive hypothesis would read "Assume for all strings x and y such that $|x| + |y| < |w| + |z|$ that $(x \bullet y)^R = x^R \bullet y^R$."*

3. Prove that $(w^R)^R = w$ for every string w.

Solution (induction on w):

Let w be an arbitrary string.

Assume for every string x where $|x| < |w|$ that $(x^R)^R = x$.

There are two cases to consider.

- If $w = \varepsilon$, then

$$\begin{aligned}
(w^R)^R &= (\varepsilon^R)^R && \text{because } w = \varepsilon \\
&= \varepsilon^R && \text{by definition of }^R \\
&= \varepsilon && \text{by definition of }^R \\
&= w && \text{because } w = \varepsilon
\end{aligned}$$

- Otherwise, $w = ax$ for some symbol a and some string x.

$$\begin{aligned}
(w^R)^R &= ((ax)^R)^R && \text{because } w = ax \\
&= (x^R \bullet a)^R && \text{by definition of }^R \\
&= a^R \bullet (x^R)^R && \textbf{by problem 2} \\
&= (\varepsilon^R \bullet a) \bullet (x^R)^R && \text{by definition of }^R \\
&= (\varepsilon \bullet a) \bullet (x^R)^R && \text{by definition of }^R \\
&= a \bullet (x^R)^R && \text{by definition of } \bullet \\
&= a \cdot (x^R)^R && \text{by definition of } \bullet \\
&= a \cdot x && \text{by the induction hypothesis} \\
&= w && \text{because } w = ax
\end{aligned}$$

In both cases, we conclude that $(w^R)^R = w$. ■

To think about later: Let $\#(a, w)$ denote the number of times symbol a appears in string w. For example, $\#(\texttt{X}, \texttt{WTF374}) = 0$ and $\#(\texttt{0}, \texttt{000010101010010100}) = 12$.

4. Give a formal recursive definition of $\#(a, w)$.

Solution:

$$\#(a,w) = \begin{cases} 0 & \text{if } w = \varepsilon \\ 1 + \#(a,x) & \text{if } w = ax \text{ for some string } x \\ \#(a,x) & \text{if } w = bx \text{ for some symbol } b \neq a \text{ and some string } x \end{cases}$$

■

Solution (clever notation):

$$\#(a,w) = \begin{cases} 0 & \text{if } w = \varepsilon \\ \#(a,x) + [a = b] & \text{if } w = bx \text{ for some symbol } b \text{ and some string } x \end{cases}$$

The expression $[a = b]$ in red is Iverson bracket notation. For any proposition P, the expression $[P]$ is equal to 1 is P is true and 0 is P is false. ■

5. Prove that $\#(a, w \bullet z) = \#(a, w) + \#(a, z)$ for all symbols a and all strings w and z.

Solution (induction on w):

Let a be an arbitrary symbol, and let w and z be arbitrary strings.
Assume for any string x such that $|x| < |w|$ that $\#(a, x \bullet z) = \#(a, x) + \#(a, z)$.
There are three cases to consider.

- If $w = \varepsilon$, then

$$\begin{aligned}
\#(a, w \bullet z) &= \#(a, \varepsilon \bullet z) && \text{because } w = \varepsilon \\
&= \#(a, z) && \text{by definition of } \bullet \\
&= \#(a, \varepsilon) + \#(a, z) && \text{by definition of } \# \\
&= \#(a, w) + \#(a, z) && \text{because } w = \varepsilon
\end{aligned}$$

- If $w = ax$ for some string x, then

$$\begin{aligned}
\#(a, w \bullet z) &= \#(a, ax \bullet z) && \text{because } w = ax \\
&= \#(a, a \cdot (x \bullet z)) && \text{by definition of } \bullet \\
&= 1 + \#(a, x \bullet z) && \text{by definition of } \# \\
&= 1 + \#(a, x) + \#(a, z) && \text{by the induction hypothesis} \\
&= \#(a, ax) + \#(a, z) && \text{by definition of } \# \\
&= \#(a, w) + \#(a, z) && \text{because } w = ax
\end{aligned}$$

- If $w = bx$ for some symbol $b \neq a$ and some string x, then

$$\begin{aligned}
\#(a, w \bullet z) &= \#(a, bx \bullet z) && \text{because } w = bx \\
&= \#(a, b \cdot (x \bullet z)) && \text{by definition of } \bullet \\
&= \#(a, x \bullet z) && \text{by definition of } \# \\
&= \#(a, x) + \#(a, z) && \text{by the induction hypothesis} \\
&= \#(a, bx) + \#(a, z) && \text{by definition of } \# \\
&= \#(a, w) + \#(a, z) && \text{because } w = bx
\end{aligned}$$

In every case, we conclude that $\#(a, w \bullet z) = \#(a, w) + \#(a, z)$. ■

6. Prove that $\#(a, w^R) = \#(a, w)$ for all symbols a and all strings w.

Solution (induction on w): Let a be an arbitrary symbol, and let w be an arbitrary string.

Assume for any string x such that $|x| < |w|$ that $\#(a, x^R) = \#(a, x)$.

There are three cases to consider.

- If $w = \varepsilon$, then $w^R = \varepsilon = w$ by definition, so $\#(a, w^R) = \#(a, w)$.
- If $w = ax$ for some string x, then

$$\begin{aligned}
\#(a, w^R) &= \#(a, (ax)^R) && \text{because } w = ax \\
&= \#(a, x^R \bullet a) && \text{by definition of } ^R \\
&= \#(a, x^R) + \#(a, a) && \text{by problem 5} \\
&= \#(a, x^R) + 1 && \text{by definition of } \# \\
&= \#(a, x) + 1 && \text{by the induction hypothesis} \\
&= \#(a, ax) && \text{by definition of } \# \\
&= \#(a, w) && \text{because } w = ax
\end{aligned}$$

- If $w = bx$ for some symbol $b \neq a$ and some string x, then

$$\begin{aligned}
\#(a, w^R) &= \#(a, (bx)^R) && \text{because } w = bx \\
&= \#(a, x^R \bullet b) && \text{by definition of } ^R \\
&= \#(a, x^R) + \#(a, b) && \text{by problem 5} \\
&= \#(a, x^R) && \text{by definition of } \# \\
&= \#(a, x) && \text{by the induction hypothesis} \\
&= \#(a, bx) && \text{by definition of } \# \\
&= \#(a, w) && \text{because } w = ax
\end{aligned}$$

In every case, we conclude that $\#(a, w^R) = \#(a, w)$. ■