

1. Describe an algorithm to compute the maximum total score you can achieve. The input to your sweet algorithm is the pair of arrays  $Score[1..n]$  and  $Wait[1..n]$ .

**Solution:** Let  $MaxTotal(i)$  denote the maximum score you can achieve dancing songs  $i$  through  $n$ , assuming you are physically able to dance to song  $i$ . We need to compute  $MaxTotal(1)$ . The function satisfies the following recurrence:

$$MaxTotal(i) = \begin{cases} 0 & \text{if } i > n \\ \max \left\{ \begin{array}{l} MaxTotal(i + 1), \\ Score[i] + MaxTotal(i + Wait[i] + 1) \end{array} \right\} & \text{otherwise} \end{cases}$$

We can memoize this function into a one-dimensional array  $MaxTotal[1..n]$  (using the value 0 whenever we need to look up  $MaxTotal(i)$  for  $i > n$ .) We can fill this array in constant time per entry by decreasing  $i$  in the only loop. **The resulting algorithm runs in  $O(n)$  time.** ■

**Rubric:** 10 points: standard dynamic programming rubric.

2. Describe an algorithm to find the largest possible number of *good* districts in a legal partition. In a legal partition, every district has between  $k$  and  $2k$  residents; a district is good if a strict majority of its residents previously voted for Oceania. Your input consists of the integer  $k$  and a boolean array  $GoodVote[1..n]$  indicating which residents previously voted for Oceania (TRUE) or Eurasia (FALSE). You can assume that a legal partition exists. Analyze the running time of your algorithm in terms of the parameters  $n$  and  $k$ .

**Solution:** Let  $MaxGood(i)$  denote the maximum number of good districts in a legal partition of voters  $i$  through  $n$ . We need to compute  $MaxGood(1)$ . This function satisfies the following recurrence:

$$MaxGood(i) = \begin{cases} 0 & \text{if } i > n \\ -\infty & \text{if } n - i + 1 < k \\ \max \left\{ \begin{array}{l} GoodDistrict(i, j) \\ + MaxGood(j + 1) \end{array} \middle| \begin{array}{l} k \leq j - i + 1 \leq 2k \\ j \leq n \end{array} \right\} & \text{otherwise} \end{cases}$$

Here  $GoodDistrict(i, j) = 1$  if a majority of voters  $i$  through  $j$  voted for Oceania, and 0 otherwise. We can compute this value in  $O(j - i + 1) = O(k)$  time by brute force.

We can memoize this function into a one-dimensional array  $MaxGood[1..n]$ , which we can fill from right to left (decreasing  $i$ ). For each index  $i$ , we need to compute  $GoodDistrict(i, j)$  for  $k + 1$  different values of  $j$ . Each call to  $GoodDistrict(i, j)$  takes  $O(k)$  time, so the total time to compute each  $MaxGood[i]$  is  $O(k^2)$ . We conclude that our algorithm runs in  $O(nk^2)$  time.

We can speed up this algorithm by pre-computing some information. Let  $NumGood(i)$  denote the number of good voters among voters 1 through  $i$ . This function satisfies the following recurrence:

$$NumGood(i) = \begin{cases} 0 & \text{if } i = 0 \\ GoodVote[i] + NumGood(i - 1) & \text{otherwise} \end{cases}$$

We can memoize this function into an arrays  $NumGood[1..n]$ , which we can fill in reverse index order in  $O(n)$  time.

Once this array has been filled, we can compute  $GoodDistrict(i, j)$  for any  $i$  and  $j$  in  $O(1)$  time as follows:

$$GoodDistrict(i, j) = [(NumGood[j] - NumGood[i - 1]) > (j - i + 1)/2]$$

(Here  $j - i + 1$  is the number of voters, and  $NumGood[j] - NumGood[i - 1]$  is the number of *good* voters, among voters  $i$  through  $j$ .) With this optimization, each value  $MaxGood[i]$  can be computed in  $O(k)$  time, so the overall algorithm runs in  $O(nk)$  time. ■

**Rubric:** 10 points: standard dynamic programming rubric. 8 points for an  $O(nk^2)$ -time algorithm; scale partial credit. This is not the only correct solution.

3. Suppose we want to split an array  $A[1..n]$  of integers into  $k$  contiguous intervals that partition the sum of the values as evenly as possible. Specifically, define the *cost* of such a partition as the maximum, over all  $k$  intervals, of the sum of the values in that interval; our goal is to minimize this cost. Describe and analyze an algorithm to compute the minimum cost of a partition of  $A$  into  $k$  intervals, given the array  $A$  and the integer  $k$  as input.

**Solution:** We define three functions:

- $PrefSum(i)$  is the sum of all elements of the prefix  $A[1..i]$ .
- $Sum(i, j)$  is the sum of all elements of the interval  $A[i..j]$ .
- $MinCost(i, \ell)$  is the minimum cost of a partition of  $A[i..n]$  into  $\ell$  intervals.

We need to compute  $MinCost(1, k)$ . These functions satisfy the following recurrences:

$$PrefSum(i) = \begin{cases} 0 & \text{if } i = 0 \\ A[i] + PrefSum(i - 1) & \text{otherwise} \end{cases}$$

$$Sum(i, j) = PrefSum(j) - PrefSum(i - 1)$$

$$MinCost(i, \ell) = \begin{cases} Sum(i, n) & \text{if } \ell = 1 \\ \min \left\{ \max \left\{ \begin{array}{l} Sum(i, j), \\ MinCost(j + 1, \ell - 1) \end{array} \right\} \mid i \leq j \leq n \right\} & \text{otherwise} \end{cases}$$

We can memoize  $PrefSum$  into an array  $PrefSum[0..n]$ , which we can fill from left to right in  $O(n)$  time. We don't need to memoize  $Sum$ . Finally, we can memoize  $MinCost$  into an array  $MinCost[1..n, 0..k]$ , which we can fill in standard column-major order: increasing  $\ell$  in the outer loop, and considering  $i$  in any order in the inner loop.

The overall algorithm runs in  $O(n^2k)$  time. ■

*This solution assumes that partitions are allowed to contain empty intervals; without loss of generality, all the empty intervals in any partition are at the end of the array. To forbid empty intervals, modify the base cases as follows:*

$$MinCost(i, \ell) = \begin{cases} \infty & \text{if } i > n \\ Sum(i, n) & \text{if } i \leq n \text{ and } \ell = 1 \\ \min \left\{ \max \left\{ \begin{array}{l} Sum(i, j), \\ MinCost(j + 1, \ell - 1) \end{array} \right\} \mid i \leq j \leq n \right\} & \text{otherwise} \end{cases}$$

**Solution:** For any integers  $i$  and  $\ell$ , let  $Min\$(i, \ell)$  denote the minimum cost of a partition of  $A[i..n]$  into  $\ell$  intervals. We need to compute  $Min\$(1, k)$ . This function satisfies the following recurrences:

$$Min\$(i, \ell) = \begin{cases} -\infty & \text{if } \ell = 0 \\ 0 & \text{if } i > n \text{ and } \ell > 0 \\ \min \left\{ \max \left\{ \sum_{h=i}^j A[h], Min\$(j+1, \ell-1) \right\} \mid i \leq j \leq n \right\} & \text{otherwise} \end{cases}$$

In particular,  $Min\$(i, 0) = -\infty$  because the maximum of the empty set (in this case, the set containing zero interval sums) is  $-\infty$ , and  $Min\$(n+1, \ell) = 0$  when  $\ell > 0$  because the only legal partition of the empty sequence consists of empty intervals, each of which sums to 0.

We can memoize  $Min\$(i, \ell)$  into an array  $Min\$(1..n+1, 0..k)$ , which we can fill row-by-row bottom up in the outer loop, filling each row in arbitrary order in the inner loop. We can compute each entry  $Min\$(i, \ell)$  in  $O(n^2)$  time by looping over  $j$  and  $h$ . Thus, the entire algorithm runs in  $O(n^3k)$  time.

However, we can speed up this algorithm with some precomputation. For any index  $j$ , let  $PrefSum(j)$  denote the prefix sum  $\sum_{h=1}^j A[h]$ . This function satisfies the recurrence

$$PrefSum(j) = \begin{cases} 0 & \text{if } j = 0 \\ A[j] + PrefSum(j-1) & \text{otherwise} \end{cases}$$

We can memoize this function into an array  $PrefSum[0..n]$ , which we can fill from left to right in  $O(n)$  time. Then we can evaluate any sum  $\sum_{h=i}^j A[h]$  in  $O(1)$  time, because

$$\sum_{h=i}^j A[h] = PrefSum[j] - PrefSum[i-1].$$

With this optimization, our algorithm runs in  $O(n^2k)$  time. ■

**Rubric:** 10 points =

- + 8 for  $O(n^3k)$ -time dynamic programming algorithm (standard DP rubric, scaled)
- + 2 for  $O(n^2k)$ -time algorithm, either by memoizing  $Sum$  (either directly in  $O(n^2)$  time or via prefix sums in  $O(n)$  time as above), or by computing  $Sum(i, j)$  on the fly in the inner loop for  $MinCost$ .

We don't care whether you allow or forbid empty intervals, as long as you're consistent.

4. A sequence of integers is *mostly odd* if strictly more than half of its elements are odd. Describe an algorithm that computes the length of the longest *mostly odd* increasing subsequence of a given array  $A[1..n]$  of integers. (You can assume that  $A$  has at least one mostly-odd increasing subsequence.)

**Solution:** To simplify case analysis at the end, we add a sentinel value  $A[0] = -\infty$ .

For any integers  $i, j$ , and  $k$ , let  $LISE(i, j, k)$  denote the length of the longest increasing subsequence of  $A[j..n]$  containing only numbers greater than  $A[i]$  and containing exactly  $k$  even elements. (Crucially,  $A[i]$  is not included in either the subsequence length or the count of even elements.) This function satisfies the following recurrence:

$$LISE(i, j, k) = \begin{cases} -\infty & \text{if } k < 0 \\ 0 & \text{if } j > n \text{ and } k = 0 \\ -\infty & \text{if } j > n \text{ and } k > 0 \\ LISE(i, j + 1, k) & \text{if } A[i] \geq A[j] \\ \max \left\{ \begin{array}{l} LISE(i, j + 1, k) \\ 1 + LISE(j, j + 1, k) \end{array} \right\} & \text{if } A[i] < A[j] \text{ and } A[j] \text{ is odd} \\ \max \left\{ \begin{array}{l} LISE(i, j + 1, k) \\ 1 + LISE(j, j + 1, k - 1) \end{array} \right\} & \text{otherwise} \end{cases}$$

We can memoize this function into a three-dimensional array  $LISE[0..n, 1..n, 1..n/2]$ , which we can fill with three nested for-loops, decreasing  $j$  in the outermost loop and considering  $i$  and  $k$  in any order in the two inner loops.

Finally, we find and return

$$\max \{ LISE[0, 1, k] \mid LISE[0, 1, k] \geq 2k + 1 \}.$$

The entire algorithm runs in  $O(n^3)$  time. ■

**Rubric:** 10 points, standard dynamic programming rubric. This is not the only correct solution. I do not know whether this is the fastest algorithm for this problem.