The following problems ask you to prove some "obvious" claims about recursively-defined string functions. In each case, we want a self-contained, step-by-step induction proof that builds on formal definitions and prior results, *not* on intuition. In particular, your proofs must refer to the formal recursive definitions of string length and string concatenation:

$$|w| := \begin{cases} 0 & \text{if } w = \varepsilon \\ 1 + |x| & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

$$w \bullet z := \begin{cases} z & \text{if } w = \varepsilon \\ a \cdot (x \bullet z) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

You may freely use the following results:

Lemma 1: $w \bullet \varepsilon = w$ for all strings w.

Lemma 2: $|w \bullet z| = |w| + |z|$ for all strings w and z.

Lemma 3: $(w \bullet y) \bullet z = w \bullet (y \bullet z)$ for all strings w, y, and z.

Inductive proofs of these lemmas (extracted directly from the lecture notes) appear on the following pages. Your inductive proofs should follow the general structure of these examples.

The ***reversal*** $\boldsymbol{w^R}$ of a string w is defined recursively as follows:

$$w^R := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ x^R \bullet a & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

For example, $\texttt{STRESSED}^R = \texttt{DESSERTS}$ and $\texttt{WTF374}^R = \texttt{473FTW}$.

1. Prove that $|w| = |w^R|$ for every string w.
2. Prove that $(w \bullet z)^R = z^R \bullet w^R$ for all strings w and z.
3. Prove that $(w^R)^R = w$ for every string w.

[Hint: The proof for problem 3 relies on problem 2, but it may be easier to solve problem 3 first.]

To think about later: Let $\#(a, w)$ denote the number of times symbol a appears in string w. For example, $\#(\texttt{X}, \texttt{WTF374}) = 0$ and $\#(\texttt{0}, \texttt{000010101010010100}) = 12$.

4. Give a formal recursive definition of $\#(a, w)$. (Your definition should have the same format as the definitions of $|w|$ and $w \bullet z$ at the top of this page.)
5. Prove that $\#(a, w \bullet z) = \#(a, w) + \#(a, z)$ for all symbols a and all strings w and z.
6. Prove that $\#(a, w^R) = \#(a, w)$ for all symbols a and all strings w.

Lemma 1. *$w \bullet \varepsilon = w$ for every string w.*

Proof: *Let w be an arbitrary string.*

Assume that $\boldsymbol{x \bullet \varepsilon = x}$ *for every string x such that $|x| < |w|$.*

There are two cases to consider:

- *Suppose $w = \varepsilon$.*

$$\begin{aligned} w \bullet \varepsilon &= \varepsilon \bullet \varepsilon && \text{because } w = \varepsilon, \\ &= \varepsilon && \text{by definition of } \bullet, \\ &= w && \text{because } w = \varepsilon. \end{aligned}$$

- *Suppose $w = ax$ for some symbol a and string x.*

$$\begin{aligned} w \bullet \varepsilon &= (a \cdot x) \bullet \varepsilon && \text{because } w = ax, \\ &= a \cdot (x \bullet \varepsilon) && \text{by definition of } \bullet, \\ &= a \cdot x && \text{by the inductive hypothesis,} \\ &= w && \text{because } w = ax. \end{aligned}$$

In both cases,

we conclude that $\boldsymbol{w \bullet \varepsilon = w}$. □

The nested boxes above try to emphasize this proof's *structure*. The *green italic* text is boilerplate for almost all string-induction proofs. The case breakdown directly mirror cases in the recursive definitions of strings and concatenation. The ***red bold*** text is the meat of the induction hypothesis and the result we're trying to prove. I'll use the same coloring in later proofs, but I'll omit the boxes.

We strongly recommend *writing* induction proofs "outside in": Write all the boilerplate text in the larger boxes before thinking about what to write in smaller boxes. We also recommend writing the most general ("inductive") cases before thinking about special ("base") cases, and writing the derivation for each case from both ends toward the middle.

Lemma 2. $|w \bullet z| = |w| + |z|$ *for all strings w and z.*

Proof: *Let w and z be arbitrary strings.*
Assume that $|x \bullet z| = |x| + |z|$ *for every string y such that $|x| < |w|$.*
(Notice that we are inducting only on w.)
There are two cases to consider:

- *Suppose $w = \varepsilon$.*

$$
\begin{aligned}
|w \bullet z| &= |\varepsilon \bullet z| && \text{because } w = \varepsilon \\
&= |z| && \text{by definition of } \bullet \\
&= 0 + |z| && \text{by definition of } + \\
&= |\varepsilon| + |z| && \text{by definition of } |\cdot| \\
&= |w| + |z| && \text{because } w = \varepsilon
\end{aligned}
$$

- *Suppose $w = ax$ for some symbol a and string x.*

$$
\begin{aligned}
|w \bullet z| &= |ax \bullet z| && \text{because } w = ax \\
&= |a \cdot (x \bullet z)| && \text{by definition of } \bullet \\
&= 1 + |x \bullet z| && \text{by definition of } |\cdot| \\
&= 1 + |x| + |z| && \text{by the inductive hypothesis} \\
&= |ax| + |z| && \text{by definition of } |\cdot| \\
&= |w| + |z| && \text{because } w = ax
\end{aligned}
$$

In both cases, we conclude that $|w \bullet z| = |w| + |z|$. □

Lemma 3. $(w \bullet y) \bullet z = w \bullet (y \bullet z)$ *for all strings* w, y, *and* z.

Proof: *Let* w, y, *and* z *be arbitrary strings.*
Assume that $\boldsymbol{(x \bullet y) \bullet z = x \bullet (y \bullet z)}$ *for every string* x *such that* $|x| < |w|$.
(Notice again that we are inducting only on w.)
There are two cases to consider.

- *Suppose* $w = \varepsilon$.

$$\begin{aligned}(w \bullet y) \bullet z &= (\varepsilon \bullet y) \bullet z && \text{because } w = \varepsilon \\ &= y \bullet z && \text{by definition of } \bullet \\ &= \varepsilon \bullet (y \bullet z) && \text{by definition of } \bullet \\ &= w \bullet (y \bullet z) && \text{because } w = \varepsilon\end{aligned}$$

- *Suppose* $w = ax$ *for some symbol* a *and some string* x.

$$\begin{aligned}(w \bullet y) \bullet z &= (ax \bullet y) \bullet z && \text{because } w = ax \\ &= (a \cdot (x \bullet y)) \bullet z && \text{by definition of } \bullet \\ &= a \cdot ((x \bullet y) \bullet z) && \text{by definition of } \bullet \\ &= a \cdot (x \bullet (y \bullet z)) && \text{by the inductive hypothesis} \\ &= ax \bullet (y \bullet z) && \text{by definition of } \bullet \\ &= w \bullet (y \bullet z) && \text{because } w = ax\end{aligned}$$

In both cases, we conclude that $\boldsymbol{(w \bullet y) \bullet z = w \bullet (y \bullet z)}$. □