

Write your answers in the separate answer booklet.

You have 120 minutes (after you get the answer booklet) to answer five questions.

Please return this question sheet and your cheat sheet with your answers.

1. For each statement below, check “Yes” if the statement is *always* true and check “No” otherwise. In either case, give a brief (one short sentence) explanation of your answer. Read these statements very carefully—small details matter!
 - (a) There exists a DFA for every finite language.
 - (b) If L is regular, then $L \cap L'$ is regular for any language L' .
 - (c) If L is regular, then any $L' \subseteq L$ is regular.
 - (d) If L has a fooling set of size 374, then every DFA for L requires at least 374 states.
 - (e) If L has a fooling set of size 374, then there exists a DFA for L with 374 states.
 - (f) If F is a fooling set of a context-free language L , then F contains an infinite length string.
 - (g) For language $\{0^a 1^b 0^a \mid b > 0\}$, there exists a fooling set F such that F is regular.
 - (h) For language $\{0^a 1^b 0^a \mid b > 0\}$, every fooling set is regular.
 - (i) Let the *smallest* DFAs for languages L_1 and L_2 have k_1 and k_2 many states respectively, then the smallest DFA for language $L_1 \cap L_2$ has $(k_1 * k_2)$ states. Here, by *smallest* we mean with the fewest number of states.
 - (j) If language L is regular then language $L' = \{y \mid xy \in L, |x| \geq 374\}$ is regular.

2. For two characters $a, b \in \{0, 1\} =: \Sigma$, recall the NAND operation $a \bar{\wedge} b$:

$$\begin{aligned} a \bar{\wedge} b &= 0 && \text{if } a = b = 1 \\ a \bar{\wedge} b &= 1 && \text{if } a = 0 \text{ or } b = 0 \end{aligned}$$

For an even length string $w \in \Sigma^*$, we define $\text{Moderate}(w)$ as follows:

$$\text{Moderate}(w) := \begin{cases} \epsilon & \text{if } w = \epsilon \\ a \bar{\wedge} b \cdot \text{Moderate}(x) & \text{if } w = abx \text{ for } a, b \in \Sigma \text{ and } x \in \Sigma^* \end{cases}$$

For example, $\text{Moderate}(0011) = 10$, and $\text{Moderate}(01101101) = 1101$.

- (a) Prove that if $L \subseteq \Sigma^*$ is regular, then

$$\text{UNMODERATED}(L) := \{w \in \Sigma^* \mid |w| \text{ is even and } \text{Moderate}(w) \in L\}$$

is also regular.

- (b) Prove that if $L \subseteq \Sigma^*$ is regular, then

$$\text{MODERATED}(L) := \{\text{Moderate}(w) \mid |w| \text{ is even and } w \in L\}$$

is also regular.

3. For each of the following languages over the alphabet $\Sigma = \{0, 1, 2\}$, describe both a regular expression that matches the language and a DFA that accepts the language. You do not need to prove that your answers are correct.
- (a) All strings in Σ^* that start with **222** and have **210** as a substring.
This language contains the strings ~~2220201~~ **2220210**, **2221000**, and **222100221001**, but it does not contain the empty string, **02220210**, or **2220010201**.
- (b) All strings in Σ^* where each run of **2s** contains the substring **22** an odd number of times and is followed by a run of **1s**.
This language contains the strings **1101**, **0022110**, **1221022221**, and the empty string, but it does not contain the strings **2201**, **021221**, or **102221**.
4. Recall that $\#(a, w)$ denotes the number of occurrences of symbol a in string w . Consider the following recursively defined functions on strings over $\Sigma = \{0, 1\}$. (For a proposition P , the expression $[P]$ evaluates to 1 if P is true and 0 otherwise.)

$$\text{NumOdd1s}(w) = \begin{cases} 0 & \text{if } w = \varepsilon \\ [a = 1] & \text{if } w = a \text{ for some } a \in \Sigma \\ [a = 1] + \text{NumOdd1s}(x) & \text{if } w = abx \text{ for } a, b \in \Sigma \text{ and } x \in \Sigma^* \end{cases}$$

$$\text{NumEven1s}(w) = \begin{cases} 0 & \text{if } w = \varepsilon \\ 0 & \text{if } w = a \text{ for some } a \in \Sigma \\ [b = 1] + \text{NumEven1s}(x) & \text{if } w = abx \text{ for } a, b \in \Sigma \text{ and } x \in \Sigma^* \end{cases}$$

For example, $\text{NumOdd1s}(10111) = 3$ and $\text{NumEven1s}(10111) = 1$.

Prove that the following equality holds for all strings $w \in \{0, 1\}^*$:

$$\#(0, w) \bmod 2 = |w| + \text{NumOdd1s}(w) + \text{NumEven1s}(w) \bmod 2$$

As usual, you may assume any result proved in class, in the lecture notes, in labs, in lab solutions, or in homework solutions. In particular, you may use the fact that $\#(a, xy) = \#(a, x) + \#(a, y)$ for every symbol a and all strings x and y . Otherwise, your proof must be formal and self-contained. Do not appeal to intuition!

5. Let $L \subseteq \{0, 1, 2\}^*$ be a set of strings w in which **0s** and **2s** appear the same number of times and no prefix of w contains strictly more **2s** than **0s**. For example, L contains the strings **012102**, **00220112**, and the empty string, but L does not contain the strings **002**, **21011**, or **012201**.
- (a) **Prove** that L is not a regular language.
- (b) Describe a context-free grammar for L .