Write your answers in the separate answer booklet.
You have 120 minutes (after you get the answer booklet) to answer five questions.
Please return this question sheet and your cheat sheet with your answers.

1. For each statement below, check "Yes" if the statement is ***always*** true and check "No" otherwise. In either case, give a brief (one short sentence) explanation of your answer. Read these statements very carefully—small details matter!

   (a) There exists a DFA for the following language: All palindromes with length at most 374.

   (b) For every language L, if L is not regular, then its complement $\Sigma^* \setminus L$ is regular.

   (c) For every language L, if L^* is context-free then L^* is regular.

   (d) For every context-free language L, L^* is regular.

   (e) If L has a fooling set of size 374, then every DFA for L requires at least 374 states.

   (f) If F is a fooling set of a language L then F contains an infinite length string.

   (g) For language $\{0^a 1^b \mid a = b\}$, there exists a fooling set F such that F is regular.

   (h) For language $\{0^a 1^b \mid a = b\}$, every fooling set is regular.

   (i) If L_1 has a DFA with k_1 states, and L_2 has a DFA with k_2 states, then every DFA of language $L_1 \cup L_2$ has at most $(k_1 * k_2)$ states.

   (j) If language L is regular then language $L' = \{xy \mid x, y \in L, |x| = |y|\}$ is regular.

2. For two characters $a, b \in \{0, 1\} =: \Sigma$, recall the XOR operation $a \oplus b$:

$$\begin{aligned} a \oplus b &= 0 && \text{if } a = b \\ a \oplus b &= 1 && \text{if } a \neq b \end{aligned}$$

   For an even length string $w \in \Sigma^*$, we define $\mathsf{Compress}(w)$ to be the function that divides w into pairs of symbols and replaces each pair ab with $a \oplus b$. Formally:

$$\mathsf{Compress}(w) := \begin{cases} \epsilon & \text{if } w = \epsilon \\ (a \oplus b) \cdot \mathsf{Compress}(x) & \text{if } w = abx \text{ for } a, b \in \Sigma \text{ and } x \in \Sigma^* \end{cases}$$

   For example, $\mathsf{Compress}(111111) = 000$ and $\mathsf{Compress}(00011110) = 0101$.

   (a) ***Prove*** that if $L \subseteq \Sigma^*$ is regular, then

$$\textsc{Compressed}(L) := \{\mathsf{Compress}(w) : |w| \text{ is even and } w \in L\}$$

   is also regular.

   (b) ***Prove*** that if $L \subseteq \Sigma^*$ is regular, then

$$\textsc{UnCompressed}(L) := \{w \in \Sigma^* : |w| \text{ is even and } \mathsf{Compress}(w) \in L\}$$

   is also regular.

3. For each of the following languages over the alphabet $\Sigma = \{0, 1, 2\}$, describe both a regular expression that matches the language and a DFA that accepts the language. You do not need to prove that your answers are correct.

   (a) All strings in Σ^* that do not have 22 as a substring.

   This language contains the strings 0201, 00112110102, 2110200, and the empty string, but it does not contain the strings 001221, or 101022222.

   (b) All strings in Σ^* where each run of 2s contains the substring 22 an even number of times and is preceded by a run of 1s.

   This language contains the strings 1101, 001122211012, 0121122222, and the empty string, but it does not contain the strings 2101, 001221, or 1010222.

4. Recall that $\#(a, w)$ denotes the number of occurrences of symbol a in string w. Consider the following recursively defined functions on strings over $\Sigma = \{0, 1\}$. (For a proposition P, the expression $[P]$ evaluates to 1 if P is true and 0 otherwise.)

$$\mathsf{NumOdd1s}(w) = \begin{cases} 0 & \text{if } w = \varepsilon \\ [a = 1] & \text{if } w = a \text{ for some } a \in \Sigma \\ [a = 1] + \mathsf{NumOdd1s}(x) & \text{if } w = abx \text{ for } a, b \in \Sigma \text{ and } x \in \Sigma^* \end{cases}$$

$$\mathsf{NumEven1s}(w) = \begin{cases} 0 & \text{if } w = \varepsilon \\ 0 & \text{if } w = a \text{ for some } a \in \Sigma \\ [b = 1] + \mathsf{NumEven1s}(x) & \text{if } w = abx \text{ for } a, b \in \Sigma \text{ and } x \in \Sigma^* \end{cases}$$

   For example, $\mathsf{NumOdd1s}(10111) = 3$ and $\mathsf{NumEven1s}(10111) = 1$.

   Prove that the following equality holds for all strings $w \in \{0, 1\}^*$:

$$\#(1, w) \mod 2 = (\mathsf{NumOdd1s}(w) - \mathsf{NumEven1s}(w)) \mod 2$$

   As usual, you may assume any result proved in class, in the lecture notes, in labs, in lab solutions, or in homework solutions. In particular, you may use the fact that $\#(a, xy) = \#(a, x) + \#(a, y)$ for every symbol a and all strings x and y. Otherwise, your proof must be formal and self-contained. Do not appeal to intuition!

5. Let $L \subseteq \{0, 1\}^*$ be a set of strings in which 0s and 1s appear the same number of times, and substrings 01 and 10 also appear the same number of times. For example, L contains the strings 01100110 and 10010101 and the empty string, but L does not contain the strings 0101 or 01110 or 1010101.

   (a) ***Prove*** that L is not a regular language.

   (b) Describe a context-free grammar for L.