You have 120 minutes to answer five questions.

Write your answers in the separate answer booklet.

Please return this question sheet and your cheat sheet with your answers.

1. Let compress0s(w) be a function that takes a string w as input, and returns the string formed by compressing every run of 0s in w by half. Specifically, every run of $2n$ 0s is compressed to length n, and every run of $2n+1$ 0s is compressed to length $n+1$. For example:

$$\text{compress0s}(\underline{00000}11\underline{000}1) = \underline{000}11\underline{00}1$$
$$\text{compress0s}(11\underline{00000}1\underline{0}) = 11\underline{00}1\underline{0}$$
$$\text{compress0s}(11111) = 11111$$

Let L be an arbitrary regular language.

(a) ***Prove*** that $\left\{w \in \Sigma^* \;\middle|\; \text{compress0s}(w) \in L\right\}$ is regular.

(b) ***Prove*** that $\left\{\text{compress0s}(w) \;\middle|\; w \in L\right\}$ is regular.

2. Let L be the language of all strings over $\{0, 1\}$ that contain at least 374 consecutive 1s.

(a) Give a regular expression that matches L.

Use the notation R^k to denote the concatenation of k copies of the regular expression R; for example,

$$(1 + 01)^5 = (1 + 01)(1 + 01)(1 + 01)(1 + 01)(1 + 01)$$

(b) Describe a DFA whose language is L. *[Hint: Do not try to **draw** your DFA!]*

(c) ***Prove*** that any DFA whose language is L must have at least 375 states, using a fooling set argument.

3. Consider the following recursive function Bond, which doubles the length of any run of 0s in its input string.

$$\text{Bond}(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ 00 \cdot \text{Bond}(x) & \text{if } w = 0 \cdot x \text{ for some string } x \\ 1 \cdot \text{Bond}(x) & \text{if } w = 1 \cdot x \text{ for some string } x \end{cases}$$

(a) ***Prove*** that $|\text{Bond}(w)| \geq |w|$ for all strings w.

(b) ***Prove*** that $\text{Bond}(x \bullet y) = \text{Bond}(x) \bullet \text{Bond}(y)$ for all strings x and y.

As usual, you can assume any result proved in class, in the lecture notes, in labs, or in homework solutions.

4. Let L be the language $\left\{0^a 1^b 0^c \mid a = b \text{ or } a = c \text{ or } b = c\right\}$

   (a) ***Prove*** that L is *not* a regular language.

   (b) Describe a context-free grammar for L.

5. For each statement below, check "True" if the statement is always true and check "False" otherwise, and give a brief (one short sentence) explanation of your answer. Read these statements very carefully—small details matter!

   For any string $w \in (0+1)^*$, let w^C denote the *bitwise complement* of w, obtained by flipping every 0 in w to a 1, and vice versa. For example, $\varepsilon^C = \varepsilon$ and $000110^C = 111001$.

   (a) If $2 + 2 = 5$, then zero is odd.

   (b) $\{0^n 1 \mid n > 0\}$ is the only infinite fooling set for the language $\{0^n 1 0^n \mid n > 0\}$.

   (c) $\{0^n 1 0^n \mid n > 0\}$ is a context-free language.

   (d) The context-free grammar $S \rightarrow 00S \mid S11 \mid 01$ generates the language $\{0^n 1^n \mid n \geq 0\}$.

   (e) Every regular language is recognized by a DFA with exactly one accepting state.

   (f) Any language that can be decided by an NFA with ε-transitions can also be decided by an NFA without ε-transitions.

   (g) If L is a regular language over the alphabet $\{0, 1\}$, then $\{xy^C \mid x, y \in L\}$ is also regular.

   (h) If L is a regular language over the alphabet $\{0, 1\}$, then $\{ww^C \mid w \in L\}$ is also regular.

   (i) The regular expression $(00 + 11)^*$ represents the language of all strings over $\{0, 1\}$ of even length.

   (j) Let L_1 and L_2 be two regular languages. The language $(L_1 + L_2)^*$ is also regular.