You have 120 minutes to answer five questions.

Write your answers in the separate answer booklet.

Please return this question sheet and your cheat sheet with your answers.

1. For each statement below, check "Yes" if the statement is always true and check "No" otherwise, and give a brief (one short sentence) explanation of your answer. Read these statements very carefully—small details matter!

   (a) Every infinite language is regular.

   (b) The language $(0 + 1(01^*0)^*1)^*$ is not context-free.

   (c) Every subset of an irregular language is irregular.

   (d) The language $\{0^a1^b \mid a - b \text{ is divisible by } 374\}$ is regular.

   (e) If language L is not regular, then L has a finite fooling set.

   (f) If there is a DFA that rejects every string in language L, then L is regular.

   (g) If language L is accepted by an DFA with n states, then its complement $\Sigma^* \setminus L$ is also accepted by a DFA with n states.

   (h) 1^*0^* is a fooling set for the language $\{1^i0^{i+j}1^j \mid i, j \geq 0\}$.

   (i) Every regular language is accepted by a DFA with an odd number of accepting states.

   (j) The context-free grammar $S \to \varepsilon \mid 0S1S \mid 1S0S$ generates all strings in which the number of 0s equals the number of 1s.

2. Recall that a *run* in a string w is a maximal non-empty substring of w in which all symbols are equal. For any non-empty string $w \in \{0, 1\}^*$, let $\text{Delete1st}(w)$ denote the string obtained by deleting the first run in w. For example,

   $$\text{Delete1st}(\underline{111111}) = \varepsilon, \qquad \text{Delete1st}(\underline{000}110) = 110, \qquad \text{Delete1st}(\underline{1}00110) = 00110.$$

   Let L be an arbitrary regular language over the alphabet $\Sigma = \{0, 1\}$. ***Prove*** that the following languages are also regular.

   (a) $\textsc{Insert1st}(L) = \{w \in \Sigma^* \mid w \neq \varepsilon \text{ and } \text{Delete1st}(w) \in L\}$

   (b) $\textsc{Delete1st}(L) = \{\text{Delete1st}(w) \mid w \neq \varepsilon \text{ and } w \in L\}$

3. For any string $w \in \{0,1\}^*$, let squish(w) denote the string obtained by dividing w into pairs of symbols, replacing each pair with 0 if the symbols are equal and 1 otherwise, and keeping the last symbol if w has odd length. We can define squish recursively as follows:

$$\text{squish}(w) := \begin{cases} w & \text{if } w = \varepsilon \text{ or } w = 0 \text{ or } w = 1 \\ 0 \cdot \text{squish}(x) & \text{if } w = 00x \text{ or } w = 11x \text{ for some string } x \\ 1 \cdot \text{squish}(x) & \text{if } w = 01x \text{ or } w = 10x \text{ for some string } x \end{cases}$$

For example,

$$\text{squish}(\underbrace{00}\underbrace{10}\underbrace{11}\underbrace{10}\underbrace{01}1) = 010111$$

(a) ***Prove*** that $\#(1, \text{squish}(w)) \le \#(1, w)$ for every string w.

(b) ***Prove*** that $\#(1, \text{squish}(w))$ is even if and only if $\#(1, w)$ is even (or equivalently, that $\#(1, \text{squish}(w)) \bmod 2 = \#(1, w) \bmod 2$) for every string w.

As usual, you can assume any result proved in class, in the lecture notes, in labs, in lab solutions, or in homework solutions. In particular, you may use the fact that $\#(1, xy) = \#(1, x) + \#(1, y)$ for all strings x and y.

4. Let L be the set of all strings in $\{0,1\}^*$ in which every run of 0s is followed immediately by a *shorter* run of 1s. For example, the strings 0001100000111 and 11100100000111 and 11111 are in L, but the strings 00011111 and 000110000 are not.

(a) ***Prove*** that L is *not* a regular language.

(b) Describe a context-free grammar for L.

5. For each of the following languages L over the alphabet $\Sigma = \{0,1\}$, describe a DFA that accepts L **and** give a regular expression that represents L. You do not need to justify your answers.

(a) Strings that do not contain the subsequence 01110.

(b) Strings that contain at least two even-length runs of 1s.