Undecidability

Lecture 23

April 22, 2025

Part I

Formal Definitions

What is Computation?

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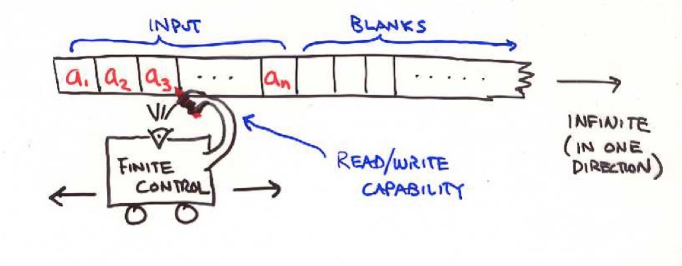

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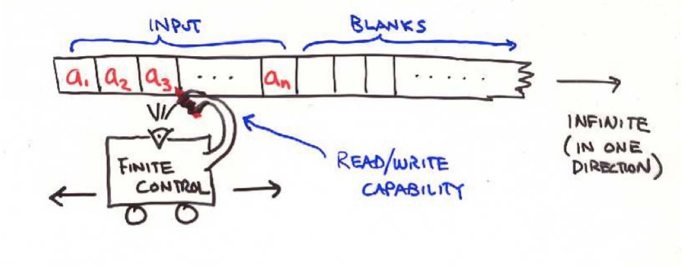

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Extremely tedious to work with TMs, so we'll just extract the properties we need and work with those.

TM Properties: Behavior

On input w, a TM either accepts w, rejects w, or runs forever.

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- $\text{Reject}(M) = \{w \in \Sigma^* \mid M \text{ rejects } w\}$
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In order for a language L to be computable (aka decidable aka recursive), we require that there is a TM M such that $\text{Accept}(M) = L$ *and* $\text{Diverge}(M) = \emptyset$.

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- If we know that $\text{Accept}(M) = L$, we call L "acceptable" or "recursively enumerable".

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Every TM can be encoded as a finite-length bit string.

- Each part of $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$ is finite, so use whatever (reasonable) encoding you want!
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- **Modifiable**: we can (algorithmically) modify an encoding to change the behavior of its TM. (For example, switch accepting with rejecting, add pre- or post-processing, etc.)
- **Executable**: There is a universal TM U that on input $\langle M, w \rangle$ simulates $M(w)$. (So $L(U) = \{\langle M, w \rangle \mid M \text{ accepts } w\}$.)

Properties of Computability

Lemma

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Lemma

Let L be an acceptable language. Then L is decidable if and only if $\Sigma^* - L$ is acceptable.

Part II

The Halting Problem

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- Each decidable language requires its own TM—of which there are only countably many!

Are there "natural" or "interesting" undecidable languages?

“Interesting” Undecidable Problems

Many problems throughout math and CS turn out to be undecidable:

- Do two CFGs generate the same language?
- What is the shortest program that outputs a given string?
- Does a multivariate polynomial have an integer root?
- Are two groups isomorphic?
- Can Conway’s Game of Life get from one pattern to another?
- Can a given set of tiles fill the plane?

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Our focus: problems that deal with computation / the behavior of programs.

The Halting Problem

Let $L_{HALT} = \{\langle M, w \rangle \mid M(w) \text{ halts}\}$.

How might you try to approach solving this problem?

Halting and Goldbach

```
CheckGoldbach():
    n = 4
    while True:
        flag = False
        for i from 2 to n − 2:
            if i and (n − i) are both prime:
                flag = True
        If flag is False:
            return False
        n = n + 2
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- If we could check if `CheckGoldbach` halts, we would resolve Goldbach's conjecture!

Halting is Undecidable

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Suppose for contradiction that there is some program `TestHalt` that decides L_{HALT}. We construct the following program:

```
Turing(x):
    Interpret x as the encoding of some TM M_x
    if TestHalt(⟨M_x, x⟩) returns True:
        Enter an infinte loop
    else
        return ''Done!''
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What does `Turing(⟨Turing⟩)` do?

Part III

Undecidability Reductions

Halting with No Input

Our proof that L_{HALT} is undecidable relied on the ability to pass `Turing` its own source code—what happens if we don't allow inputs?

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Theorem

$L_{HNI} = \{\langle M \rangle \mid M$ *halts given no input*$\}$ *is undecidable.*

Idea: show that if we can decide L_{HNI}, we can decide L_{HALT}.

```
from magic import TestHNI
TestHalt(⟨M, w⟩):
```

Halting with No Input

Our proof that L_{HALT} is undecidable relied on the ability to pass Turing its own source code—what happens if we don't allow inputs?

Theorem

$L_{HNI} = \{\langle M \rangle \mid M \text{ halts given no input}\}$ *is undecidable.*

Idea: show that if we can decide L_{HNI}, we can decide L_{HALT}.

```
from magic import TestHNI
TestHalt(⟨M, w⟩):
    Construct a program (machine) P() that runs M(w)
    return TestHNI(⟨P⟩)
```

Since L_{HALT} is undecidable, L_{HNI} must be as well.

- From last lecture, this also means that writing a 124 autograder is impossible!

Reducing Decision Problems

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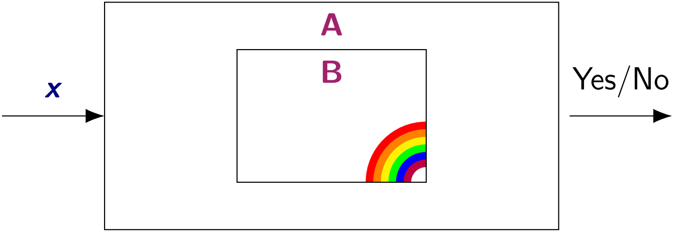
General pattern: given two *decision* problems A and B, we can reduce A to B by giving a (computable) function f such that $x \in L_A$ if and only if $f(x) \in L_B$.



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General pattern: given two *decision* problems A and B, we can reduce A to B by giving a (computable) function f such that $x \in L_A$ if and only if $f(x) \in L_B$.

When using this type of reduction, you just have to define (a computable) f and prove that $x \in L_A$ iff $f(x) \in L_B$!

- This is not the *only* way to do a reduction, but it is generally the simplest, and so the one we will use the most.

Accepting 0^{374}

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- What f has $\langle M, w \rangle \in L_{HALT}$ iff $f(\langle M, w \rangle) \in L_{374}$?

Accepting Regular Language

Let $L_{AccReg} = \{\langle M \rangle \mid \text{Accept}(M) \text{ is regular}\}$.

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- What f has $\langle M \rangle \in L_{HNI}$ iff $f(\langle M \rangle) \in L_{AccReg}$?

Accepting The Same Language

Let $L_{AccSame} = \{\langle M_1, M_2\rangle \mid \text{Accept}(M_1) = \text{Accept}(M_2)\}$.

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What language should we reduce to $L_{AccSame}$? What function f should our reduction use?

Takeaway Points

Determining if a program halts is *impossible* to do algorithmically.

- If such an algorithm existed, we could use it to construct a program that the algorithm is wrong about!

We can use this to prove that "most" problems about the behavior of a program are undecidable.

- If we can reduce an undecidable problem (eg, L_{HALT}) to our problem, we know that our problem is also undecidable.
- Simplest type of reduction: give a (computable) function f such that $\langle M, w \rangle \in L_{HALT}$ iff $f(\langle M, w \rangle)$ is in our language.