Reductions

Lecture 22

April 17, 2025

Course Outline

- Part I: models of computation (reg exps, DFA/NFA, CFGs, TMs)
- Part II: (efficient) algorithm design
- Part III: limits of (efficient) computation
    - Undecidablity: problems that have *no* algorithms
    - NP-Completeness: problems that (we think) have no *efficient* algorithms

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Key tool for proving intractability: reductions!

Part I

Reductions for Algorithms

Recall: Longest Sequences

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Longest *Decreasing* Subsequence: Find the longest subsequence of $A[1..n]$ such that each term is *smaller* than the last.

LDS Reduction

```
from magic import LIS
LDS(A[1..n]):
    Negate every element of A
    Compute seq = LIS(A)
    Negate every element of seq
    Return seq
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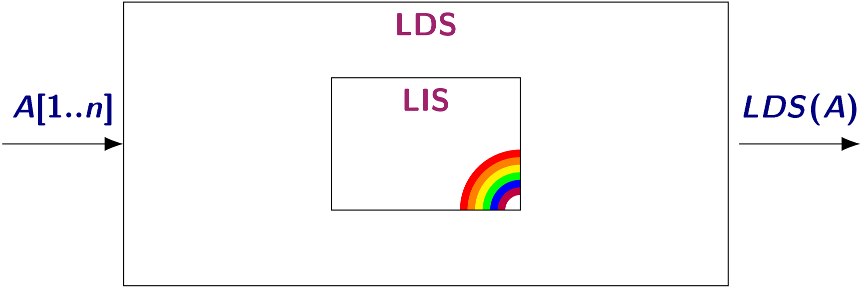

Vertex Weights

Say we have a graph $G = (V, E)$ with weights on the *vertices*. How do we find the length of the shortest path from s to t?

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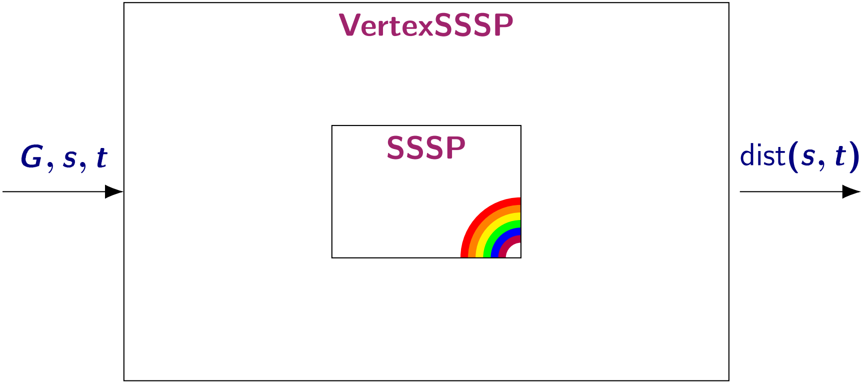

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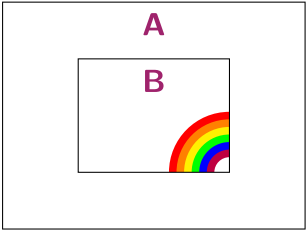

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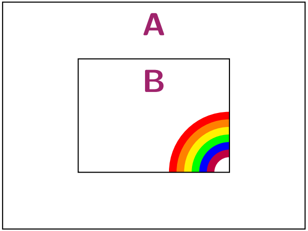

A is "no harder than" ***B***: any algorithm for ***B*** gives one for ***A***.

B is "no easier than" ***A***: if ***A*** has no "good" algorithm, neither does ***B***!

Part II

Practice with Reductions

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Autograding CS 124

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We're expecting:

```
main():
    print(‘‘Hello World!’’)
```

Can we (algorithmically) check if a student's code works?

Intuitively simpler question: can we check if a student's code at least doesn't run forever?

Reducing “Hello World!” to Halting

We want to reduce checking if a student’s code prints “Hello World!” and halts to just checking if it (eventually) halts.

```
from magic import TestHalt
TestHW(StudentCode):
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Note: ***P*** halts if and only if `StudentCode` is correct!

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This tells us that checking if a student’s code is correct is “no harder than” just checking if it runs forever.

Reducing Halting to "Hello World"

We can use these same ideas to reduce in the opposite direction!

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from magic import TestHW
TestHalt(StudentCode):
    Create a program P that:
        (1) Runs StudentCode (supressing print statements)
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Note: ***P*** will halt and print "Hello World!" if and only if `StudentCode` halts.

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Note: ***P*** will halt and print "Hello World!" if and only if `StudentCode` halts.

This means that checking if a student's code halts is "no harder than" checking if it's correct—so the two tasks are the same "level of difficulty"!

Independent Set and Clique

Given a graph $G = (V, E)$, we define

- A **clique** as $C \subseteq V$ such that each vertex in C has an edge to *every* other vertex in C.
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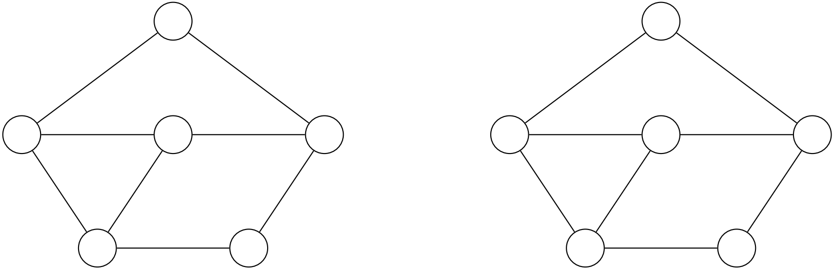

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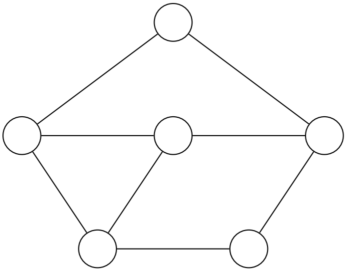
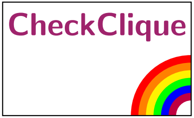
Problems of interest: given a graph G and an integer $1 \leq k \leq |V|$,

- Does G have a clique of size k?
- Does G have an independent set of size k?

Reducing Independent Set to Clique I

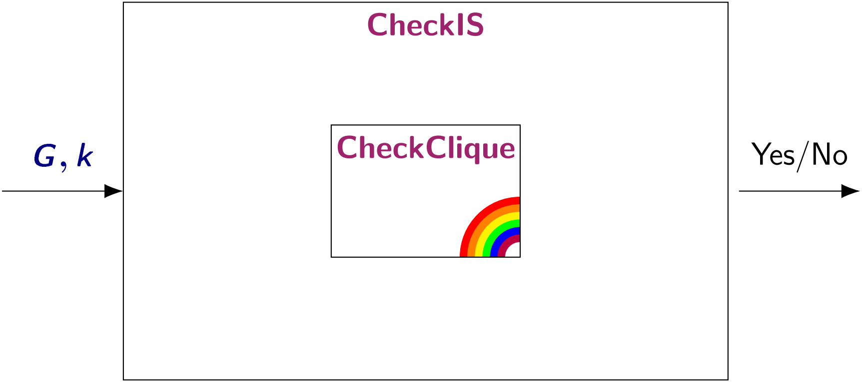
Say we wanted to check if there is an independent set of size 3 in this graph. How can we use `CheckClique` to help?





Reducing Independent Set to Clique II

We want to check if G has an independent set of size k, given the ability to check if a graph has a clique of some size.



Reducing Clique to Independent Set

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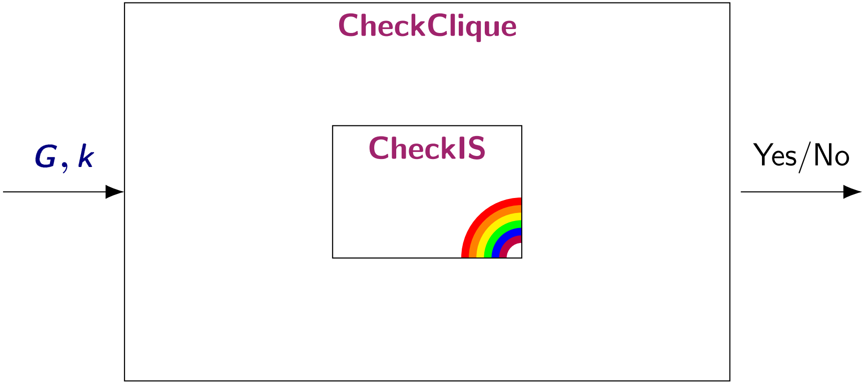

So Independent Set and Clique are "as easy / difficult" as each other!

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Can we reduce checking for a clique to checking if a program halts?

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It turns out this is impossible! (We'll see why in the next lecture.)

Takeaway: It matters which direction your reduction goes—some problems really are "strictly harder" than others!

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Concretely, how do the following problems compare?

- `CheckClique`(G, k): check if G has a clique of size k.
- `FindClique`(G, k): output a clique of size k if one exists, or say "Not possible".

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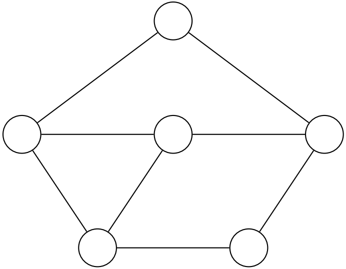
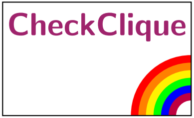
- `FindClique`(G, k): output a clique of size k if one exists, or say "Not possible".

Immediate: if we can solve `FindClique`, we can solve `CheckClique`.

Clique Search to Decision Intuition

Say we wanted to find a clique of size 3 in this graph. How can we use `CheckClique` to help?





Clique Search to Decision Reduction

```
from magic import CheckClique
FindClique(G, k):
    if CheckClique(G, k) is false:
        Return ``Not possible''
    for each vertex v ∈ G:
        Construct G' by removing v (and its edges) from G
        if CheckClique(G', k) is true:
            Remove v (and its edges) from G
    Return remaining vertices
```

Correctness?

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        if CheckClique(G′, k) is true:
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    Return remaining vertices
```

Correctness?

This means that checking if a clique exists and actually finding one "as easy / difficult" as each other!

Part III

Reductions for Decision Problems

Decision Problems

Similar to the first third of the class, we will be mostly interested in *decision* problems: our answer is either “Yes” or ”No”.

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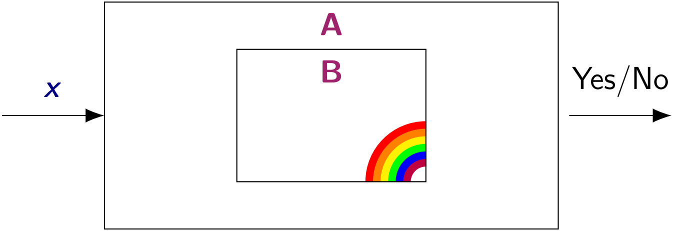
Recall that we often refer to such problems as a "language" $L \subseteq \{0, 1\}^*$—strings in L are exactly those that we want to output "Yes" on (ie, accept).

Reducing Decision Problems

Say we have two decision problems A and B. We can reduce A to B by giving a function f such that $x \in L_A$ if and only if $f(x) \in L_B$.



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Examples from before:

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When using this type of reduction, you just have to define f and prove that $x \in L_A$ iff $f(x) \in L_B$!

- This will be the most common type of reduction we use because it is the most simple.

Takeaway Points

Reductions are a powerful tool in CS

- Reducing A to a problem with a known algorithm gives us an algorithm for A.
- Reducing a "hard" problem to B tells us that B must also be "hard".

To reduce A to B, write an algorithm for A where we can use a subroutine that solves B.

- *Don't* worry about how the subroutine for B is implemented—just use that it solves B!
- For decision problems, it suffices to give a function f such that $x \in L_A$ if and only if $f(x) \in L_B$.