Greedy Algorithms

Lecture 21

April 15, 2025

Part I

An Initial Example

Recall: MSTs

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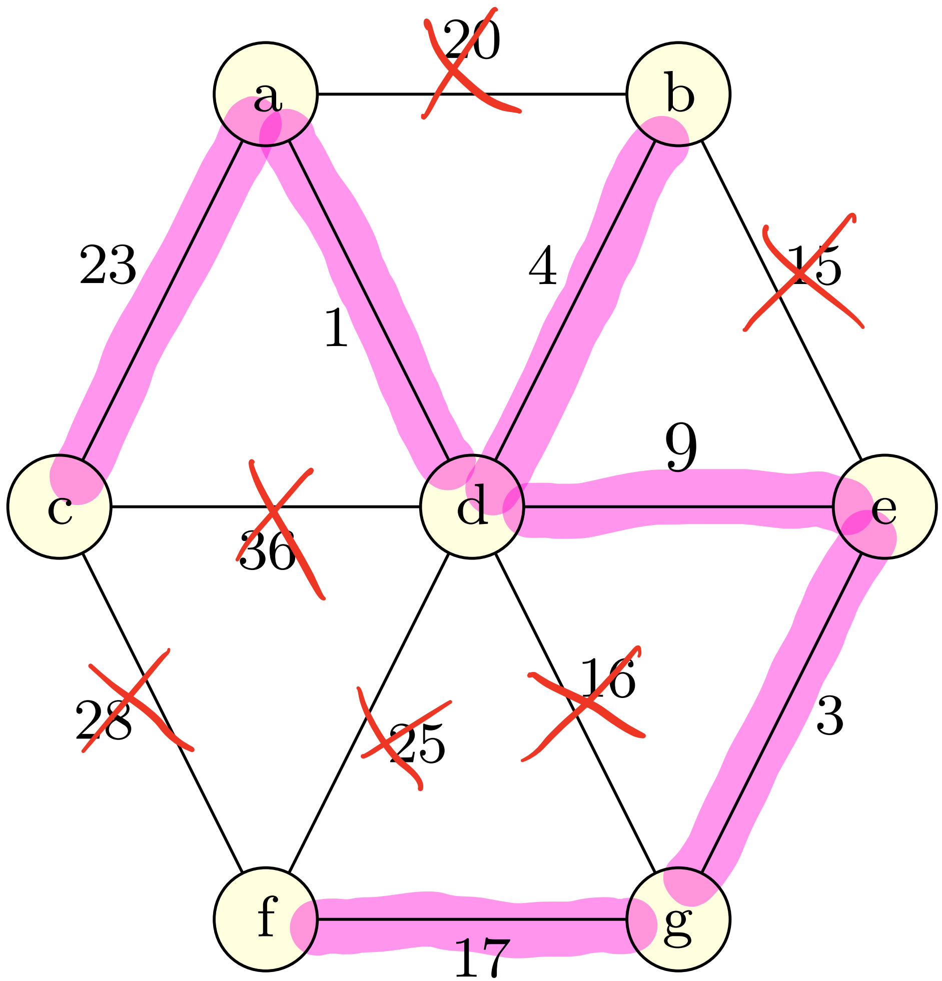
For any $S \subseteq V$, let e be the smallest edge that has exactly one endpoint in S. Then e must be in the MST.

What happens if we just repeatedly add the lightest edge anyway?

Greedy MST

Idea: add the lightest edge possible.



Kruskal's Pseudocode

```
Kruskal(G):
    Set T = ∅
    Sort edges in increasing order of weight
    for each edge e (in order by weight):
        if e connects two different components of (V, T):
            Add e to T
    return T
```

$O(E \log V)$

Correctness?

consider any edge e that Kruskal's adds to T,
e must connect two connected components: C_1, C_2
claim: e must be the lightest edge with exactly one endpoint in C_1
$\Rightarrow$ e must be in the MST

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Often argue correctness by an "exchange argument":

- Consider any solution S other than the greedy one.
- Find the "first" decision where S differs from greedy.
- Show that one can "exchange" the decision made by S for the greedy one *without making the solution worse*.

A Word of Warning

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Our advice is to always come up with an algorithm you *know* works first (eg using DP), and only then try to optimize with greedy.

Part II

Minimum Waiting Time

Problem Statement

Problem (Minimum Waiting Time)

Input: *A set of* n *jobs lengths to be scheduled on machine.*
Goal: *Order the jobs to minimize the time spent waiting, totaled over all jobs.*

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Example: say we have lengths $L = [4, 13, 10, 1, 374]$

in order: total wait time is

$0 + 4 + 17 + 27 + 28 = 76$

Greedy Attempts

Given job lengths, how do we (greedily) pick the order?

Example lengths: $L = [4, 13, 10, 1, 374]$

idea: sort jobs by increasing length

example: 1, 4, 10, 13, 374

total wait: 0 + 1 + 5 + 15 + 28 = 49

Exchange Intuition

Greedy idea: order jobs in increasing order of length.

Why is ordering **[4, 13, 10, 1, 374]** sub-optimal?

job 10: wait time decreases by 13

job 13: wait time increased by 10

every other job is unaffected!

$\Rightarrow$ improved our solution by 3.

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- Waiting time for job i ~~decreases~~ increases by $L[i+1]$
- Waiting time for job $i+1$ ~~increases~~ decreases by $L[i]$
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- Waiting time for job i decreases by $L[i+1]$
- Waiting time for job $i+1$ increases by $L[i]$
- *All* the other jobs are unaffected

Since $L[i] > L[i+1]$, this gives a strictly better solution!

(So the order we started with was not optimal.)

Part III

Job Scheduling

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Input: *A set of n jobs with start and finish times to be scheduled on a resource (eg: classes in a classroom).*

Goal: *Schedule as many jobs as possible (Two jobs with overlapping intervals cannot both be scheduled.)*

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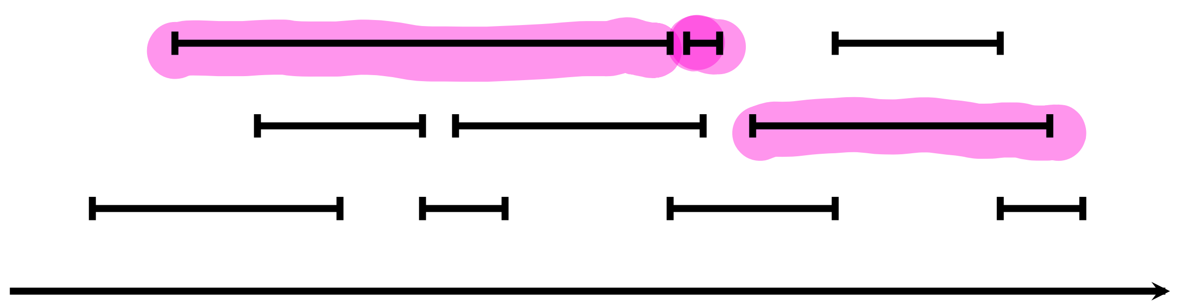
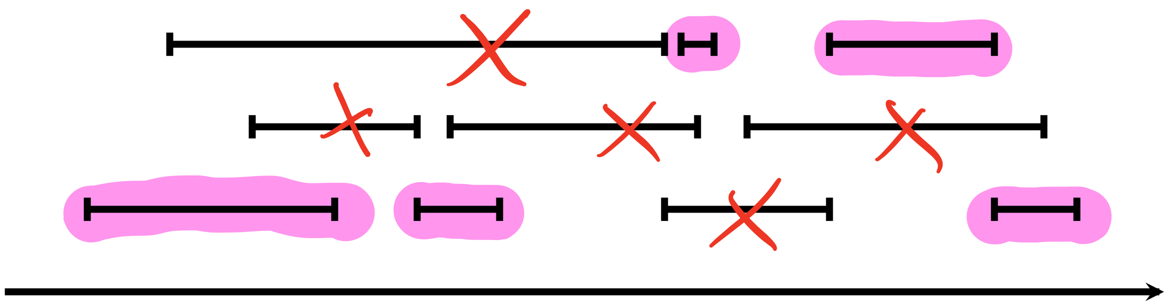
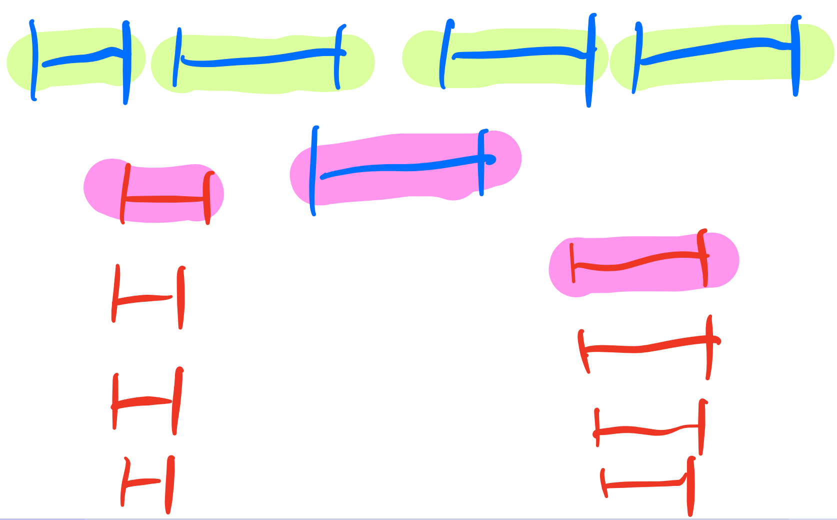


Greedy Attempts

Given job intervals, how do we (greedily) pick which to schedule?



idea: take class with fewest conflicts



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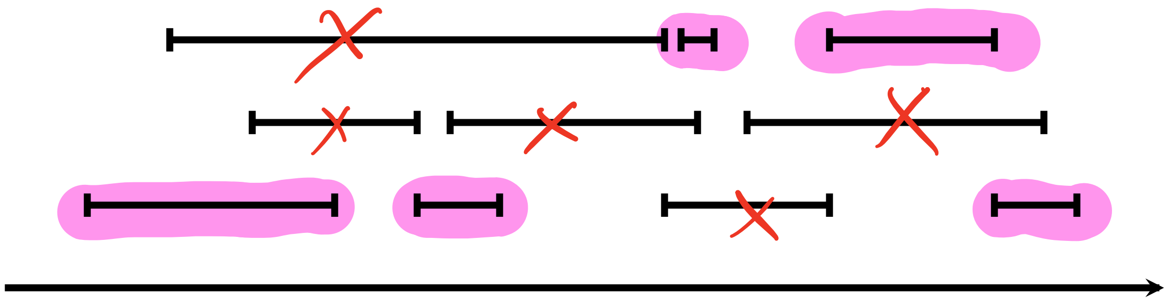


idea: take the class that ends earliest

Greedy Pseudocode

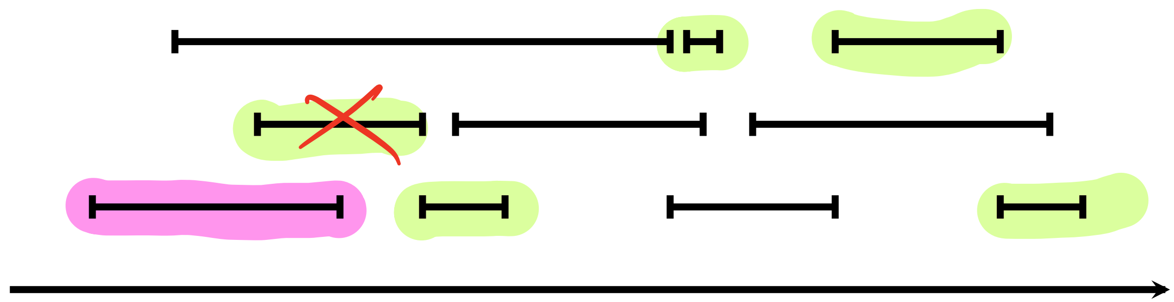
```
GreedyScheduling(J[1..n]):
    Set G = ∅ // Jobs to schedule
    Set lastAdded = −∞ // End time of last job scheduled
    Sort J by increasing end time          O(n log n)
    for each job j ∈ J:
        if j starts after lastAdded:
            Add j to G                      } → O(n)
            Set lastAdded to the end time of j
    return G
```

Efficiency? $O(n \log n)$

Exchange Intuition

Greedy idea: pick job with the earliest *finish* time.

Why can we say that any solution "may as well" use the job with the earliest finish time?



Exchange Warm-Up

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Let $g_1 = (s_1, f_1)$ *be the job with the earliest finish time. Then some optimal solution uses* g_1. *(Note: there may be multiple optimal solutions)*

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Since S has no overlaps, *all* jobs in $S - \{j_1\}$ start after $f_1' \geq f_1$.

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Since S has no overlaps, *all* jobs in $S - \{j_1\}$ start after $f'_1 \geq f_1$.

Thus $S' = (S - \{j_1\}) \cup \{g_1\}$ is a valid solution that (1) uses g_1, and (2) is optimal (since $|S'| = |S|$).

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Let $G = (g_1, g_2 \ldots, g_k)$ be the greedy solution (sorted by finish time). We can write *any* optimal solution S (sorted by finish time) as $(g_1, \ldots, g_{i-1}, j_i, \ldots, j_m)$—$g_i$ is the *first* greedy interval not in S.

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Since S has no overlaps, *all* jobs in $(j_{i+1}, \ldots, j_m)$ start after j_i finishes—which must be after g_i finishes!

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Thus $S' = (S - \{j_i\}) \cup \{g_i\}$ is an optimal solution that is "closer" to the greedy solution.

Exchange Argument II

Theorem

The greedy solution is an optimal solution.

If $S = (g_1, \ldots, g_{i-1}, j_i, \ldots, j_m)$ is an optimal solution, we can exchange j_i for g_i.

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Applying this repeatedly, we can get an optimal solution that uses the *entire* greedy solution: $S_G = (g_1, \ldots, g_k, j_{k+1}, \ldots, j_m)$.

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Note that greedy only stops when there are no intervals possible to add—so in fact we must have that $S_G = G$!

Sometimes Greed is Bad

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Takeaway: Greed is tempting, but not always the right answer.

Exercise: write a DP algorithm for the weighted version.

Takeaway Points

- Greedy algorithms have the advantage of being relatively simple to state, but often are incorrect. *Always* prove correctness.
- *Exchange* arguments are often the key proof ingredient. Start with understanding why the first step of the algorithm is correct: need to show that there is an optimum/correct solution that agrees with the first step of the greedy algorithm.
- Thinking about correctness is also a good way to figure out which of the many greedy strategies is likely to work.