DFS, Cycle Detection, and Topological Sort

Lecture 16

March 25, 2025

Recall WFS

```
WFS(G,u):
    Initialize array Found[1..n] to all FALSE
    Set Found[u] = TRUE
    Initialize ToExplore, S as {u}
    while ToExplore is non-empty do
        Remove a node x from ToExplore
        for each edge (x,y) in Adj(x) do
            if Found[y] = FALSE
                Found[y] ← TRUE
                Set y.parent = x
                Add y to ToExplore and S
    Output S
```

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BFS: Implement ToExplore with a queue (FIFO)

DFS: Implement ToExplore with a stack (LIFO)

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- Find the path from u to v that uses the fewest edges

Focus of today: uses for DFS, especially in *directed* graphs

Part I

Understanding DFS

Recursive DFS

Easier to analyze DFS if we phrase it as a recursive algorithm.

```
Global array Visited[1..n], init all False
DFS(G,u):
    Set Visited[u] = TRUE
    for each edge (u, v) in Adj(u) do
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Note: recursion stack takes the place of `ToExplore`

Pre- and Post-Ordering

Useful to know what order DFS processes vertices in, so store when DFS on u starts and finishes.

```
Global array Visited[1..n], init all False
Global integer clock, init to 1
DFS(G,u):
    u.pre = clock, clock = clock + 1
    Set Visited[u] = TRUE
    for each edge (u, v) in Adj(u) do
        if Visited[v] = FALSE
            v.parent = u
            DFS(G, v)
    u.post = clock, clock = clock + 1
```

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```

pre and post each define an ordering on the vertices.

DFSAll

To get pre- or post-ordering on the whole graph, continue to run DFS on any vertex we haven't seen yet.

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Global array Visited[1..n], init all False
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DFSAll(G):
    for each vertex u do
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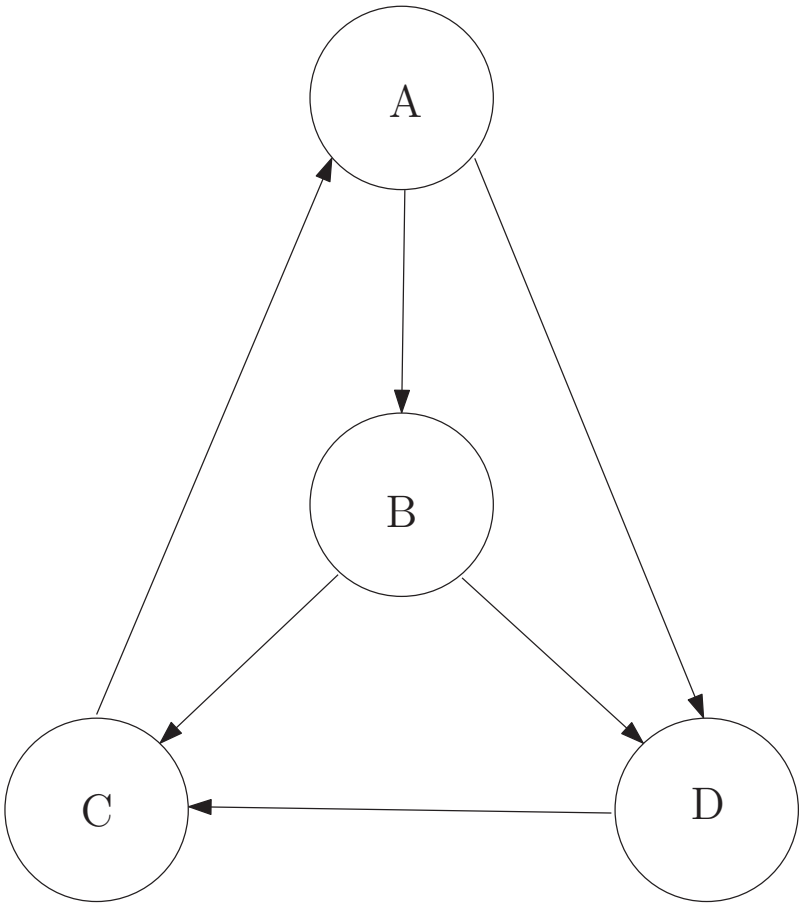
```

Example of DFS with `pre` and `post`



`clock` =

Recursion stack:

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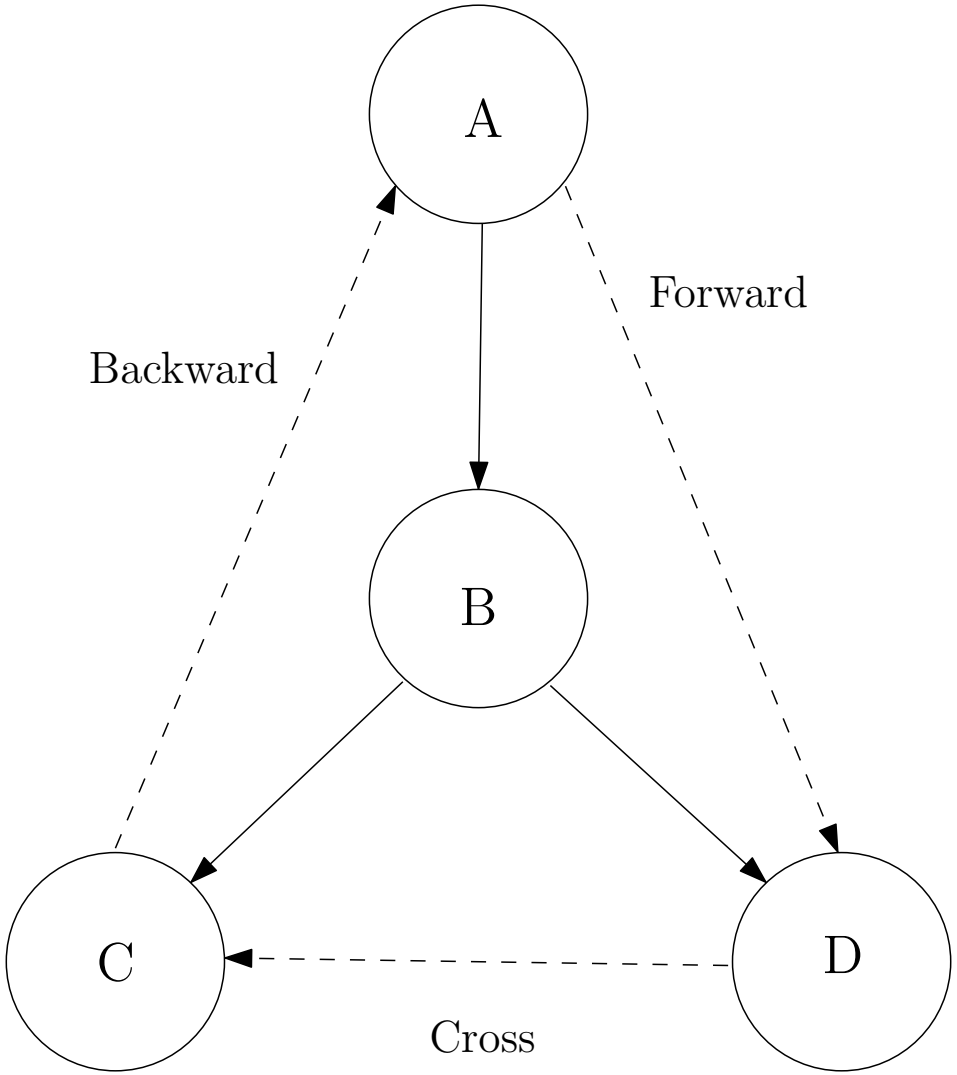
A Thousand Words



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For *any* vertices u and v, the intervals $[u.\texttt{pre}, u.\texttt{post}]$ and $[v.\texttt{pre}, v.\texttt{post}]$ are either nested or disjoint.

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For *any* vertices u and v, the intervals $[u.\texttt{pre}, u.\texttt{post}]$ and $[v.\texttt{pre}, v.\texttt{post}]$ are either nested or disjoint.

If $(u, v) \in E$, cannot have $u.\texttt{pre} < u.\texttt{post} < v.\texttt{pre} < v.\texttt{post}$.

Application: Finding Cycles

Given `pre` and `post` values from a run of DFS:

- How can we check if an *undirected* graph has a cycle?

- How can we check if a *directed* graph has a cycle?

Cycle Finding Algorithms

```
CheckUndirCycle(G):
    DFSAll(G)
    for each edge uv do
        if uv is not a tree edge
            return True
    return False
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```
CheckDirCycle(G):
    DFSAll(G)
    for each edge (u, v) do
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Efficiency?

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Efficiency? $O(V + E)$

Exercise: modify these to output a cycle if one exists.

Part II

DAGs and Topological Sort

DAG Definition

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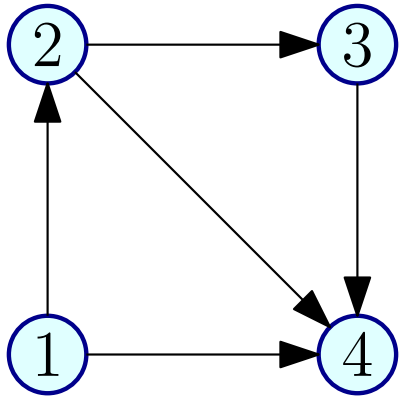

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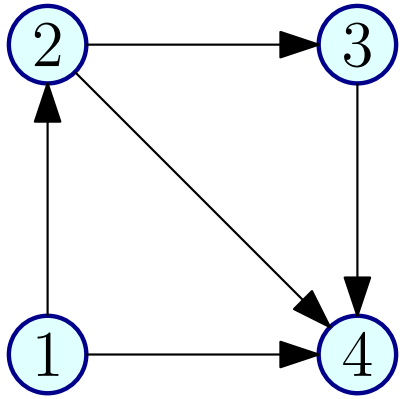

Commonly used to model dependencies or rankings.

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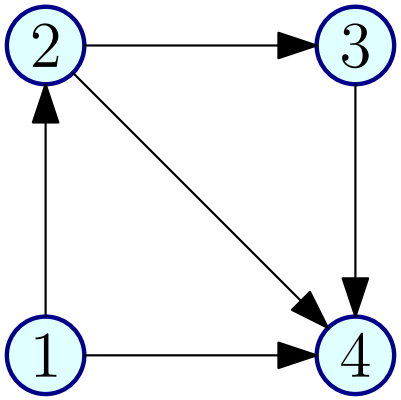

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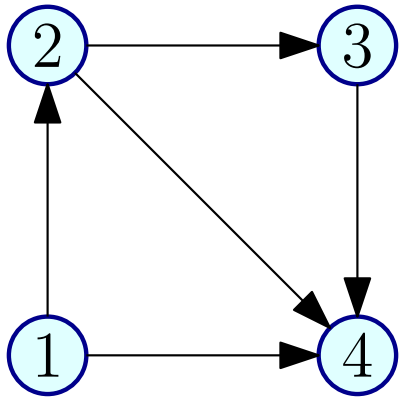

Note: it is possible for a vertex to be both a source and a sink! (Requires it to have no edges at all.)

Sources and Sinks in DAGs

Proposition

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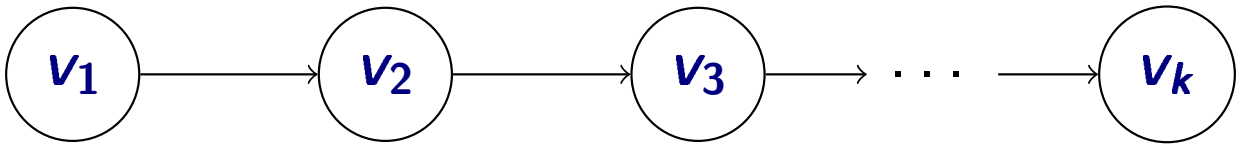
Let $(v_1, v_2, \ldots, v_k)$ be the *longest* path in our DAG.



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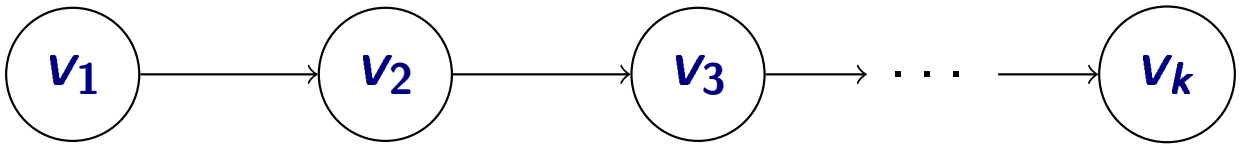

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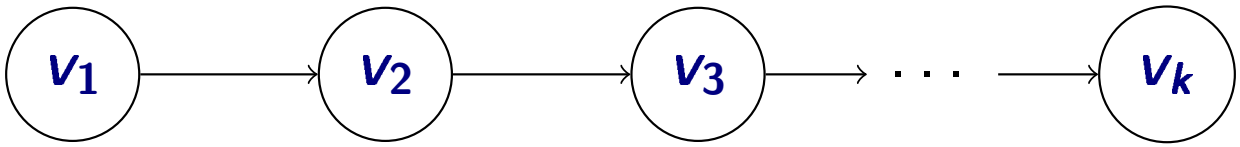

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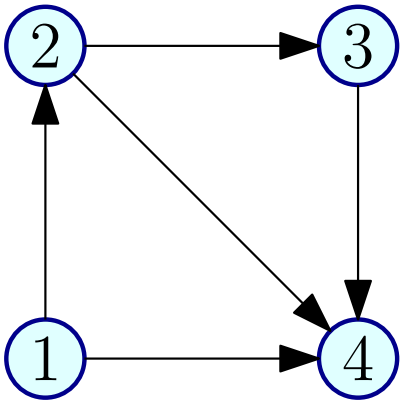
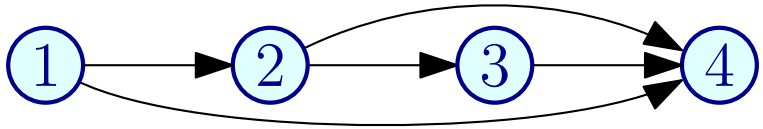


Graph G



Topological Ordering of G

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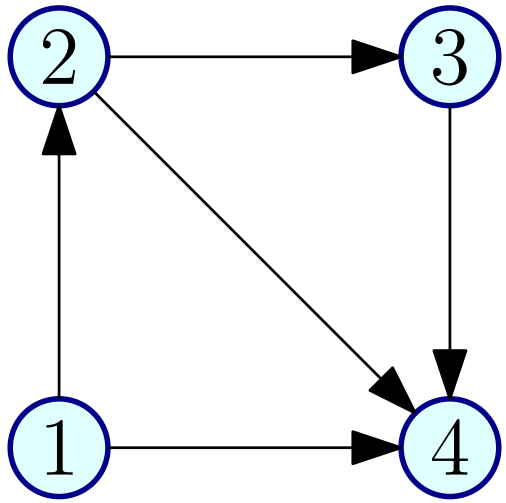
How would you find a topological order of a DAG?



Naïve Topological Order

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NaïveTopOrder(G):
    order = []
    while G is non-empty
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Efficiency? $O(V(V+E))$

Can we do better?

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```
TopOrder(G):
    DFSAll(G)
    order = []
    for all vertices v in decreasing order of v.post
        order = [v] + order
    return order
```

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If a directed graph $G = (V, E)$ has a cycle, G does not have a topological order.

In particular, this means that you should *never* invoke the topological sort algorithm on a non-DAG!

Problems and Algorithms From Today

- Cycle checking in *undirected* graphs
  - Use DFS (or WFS), $O(V + E)$ time
- Cycle checking in *directed* graphs
  - Use DFS, $O(V + E)$ time
- Finding a topological ordering in a DAG
  - Use DFS `post`-order, $O(V + E)$ time