Graph Search

Lecture 15

March 13, 2025

Part I

Graphs Review

Graph Definition

Definition

An undirected graph $G = (V, E)$ is a 2-tuple:

1. V is a set of vertices (also referred to as nodes/points)
2. E is a set of edges where each edge $e \in E$ is a set of the form $\{u, v\}$ with $u, v \in V$ and $u \neq v$.

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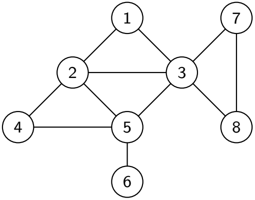

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- m or E for the number of edges $|E|$

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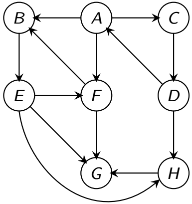

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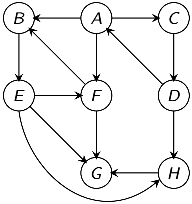

We will use both directed and undirected graphs in 374, defaulting to undirected if left unspecified.

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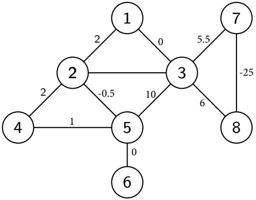

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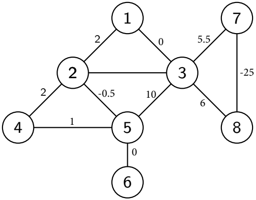

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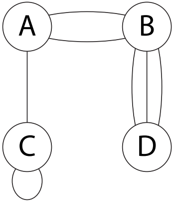

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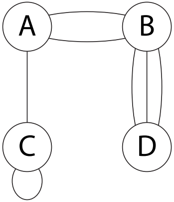

We will (almost) exclusively consider simple graphs in 374.

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- *Simple* enough to admit nice problem statements, proofs, and algorithms.
- *Abstract* enough to allow us to phrase problems we care about in terms of a graph problem, filtering out unneeded details.

Graph Example: Bridges of Köningsberg

In the map below, can we find a route that crosses each bridge *exactly* once?

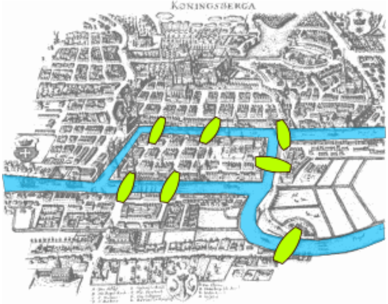
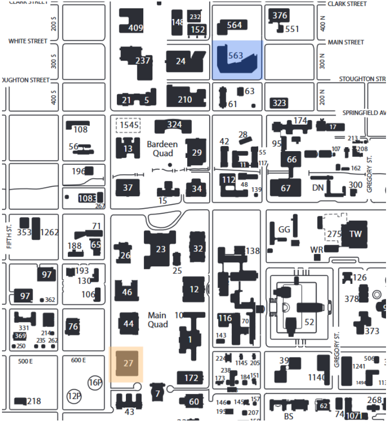


Image source: Wikipedia

Graph Example: Walking to Class

What's the fastest route to walk from Siebel to Lincoln Hall?



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- Lecture: See common graph problems and efficient algorithms to solve them.
- Lab / Homework / Exams: Given a problem (not necessarily phrased in terms of graphs), reduce it to a problem on graphs and apply a known algorithm to solve it.
- "In the wild": Want to use someone else's (optimized, bug-free) implementation of a graph algorithm to solve your particular problem of interest.

Part II

Graph Data Structures

Adjacency List

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Efficiency for graph with n vertices and m edges?

- Space
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By default, we'll assume that input graphs are represented by adjacency lists, without either of the optimizations.

- If you want to change to a different representation, make sure your run time analysis accounts for that!

Part III

Connectivity in Undirected Graphs

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For a graph $G = (V, E)$:

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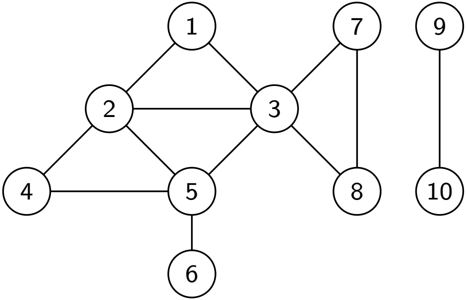
  - The requirement that $k \geq 3$ ensures that a single edge does not count as a cycle.
- A vertex u is connected to v if there is a path with $v_1 = u$ and $v_k = v$.

Definitions Examples



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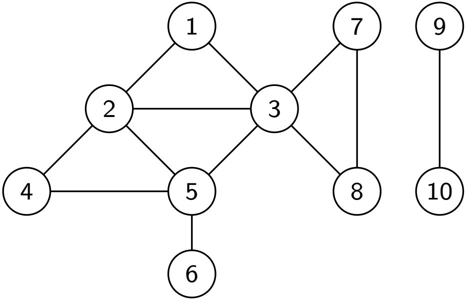

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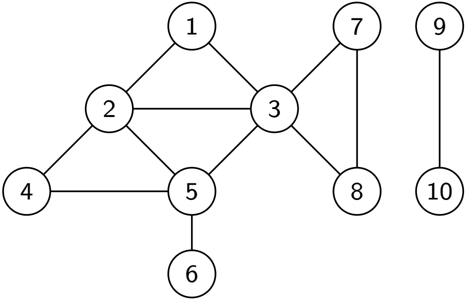

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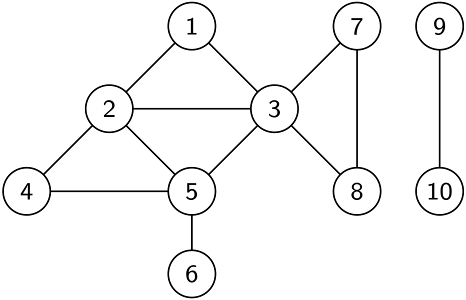

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1 is *not* connected to **10**.

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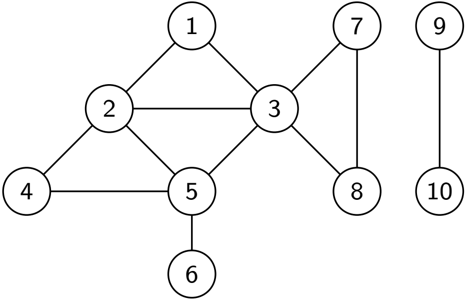
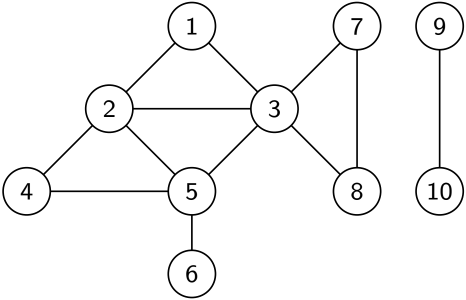


Algorithmic Connectivity

Let $G = (V, E)$ be an undirected graph. We want to solve the following problems:

1. Given nodes u and v, is u connected to v?
2. Given a node u, find the connected component containing u.
3. Find all connected components in G.

Intuitive Example



How would you find the connected component containing 6?

Naïve Search

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Naïve(G, u):
    S ← {u}
    while (there is edge uv with u ∈ S, v ∉ S) do
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Run time? $O(n \cdot m)$
Can we do better?

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    while ToExplore is non-empty
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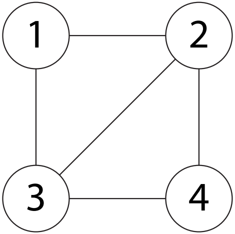
- Queue (FIFO): Breadth-first search

BFS and DFS Examples

DFS



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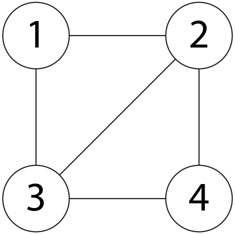
BFS



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The tree for **DFS** has some more subtle properties that we'll explore in the next lecture.