CS/ECE 374 A: Algorithms & Models of Computation

Graph Search

Lecture 15

March 13, 2025

Part I

Graphs Review

Graph Definition

Definition

An undirected graph $G = (V, E)$ is a 2-tuple:

1. V is a set of vertices (also referred to as nodes/points)
2. E is a set of edges where each edge $e \in E$ is a set of the form $\{u, v\}$ with $u, v \in V$ and $u \neq v$.

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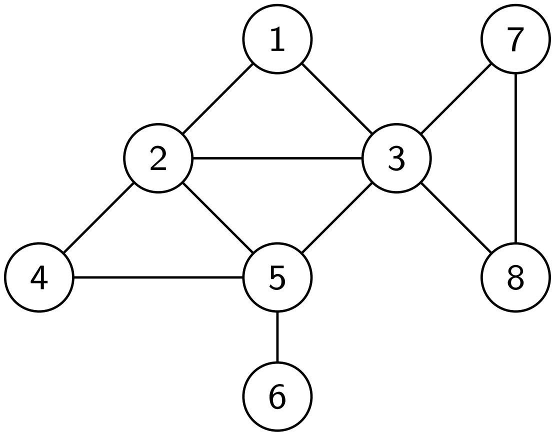

$V = \{1, 2, 3, \ldots, 8\}$

$E = \{\{1, 2\}, \{1, 3\}, \ldots\}$

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- m or E for the number of edges $|E|$

Flavors of Graphs: Directed vs Undirected

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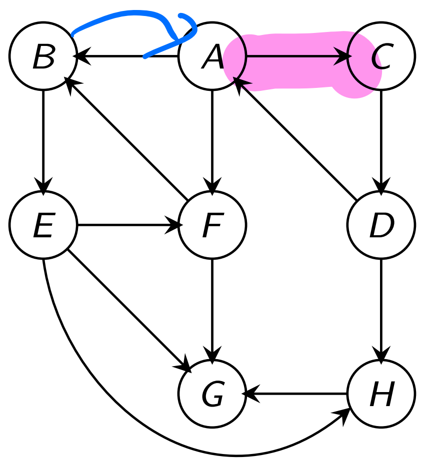

$V = \{A, B, C, \ldots\}$

$E = \{(A, C), (A, B), \ldots\}$

$(C, A) \notin E$

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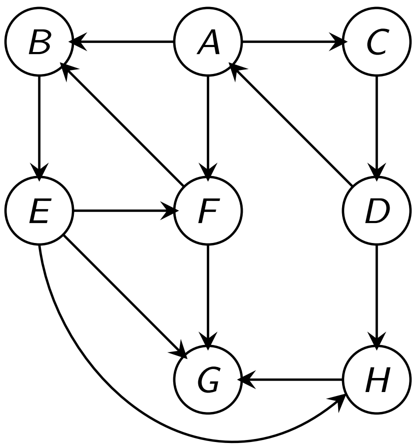

We will use both directed and undirected graphs in 374, defaulting to undirected if left unspecified.

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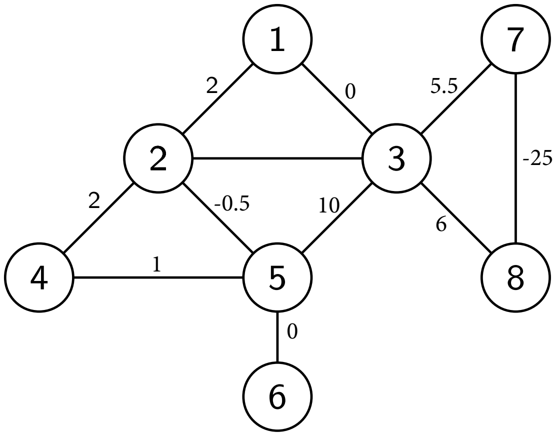

$w(12) = 2$

$w(78) = -25$

$\vdots$

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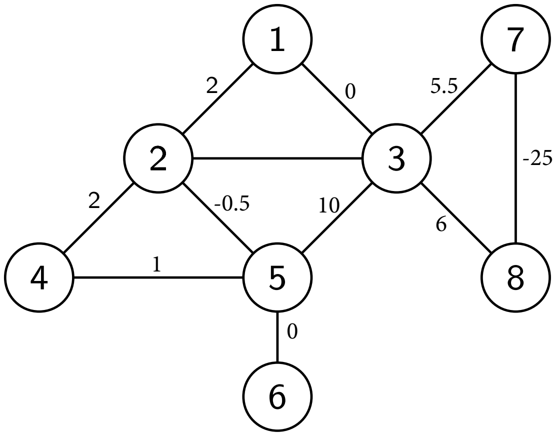

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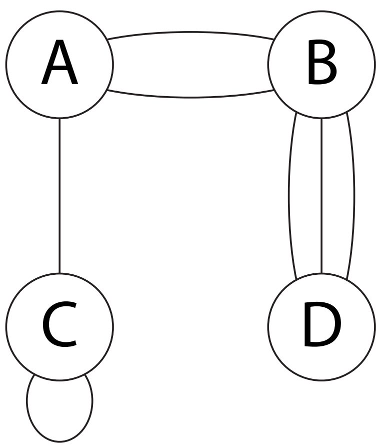

$V = \{A, B, C, D\}$

E will contain 3 copies of BD as well as 1 copy of $\{C, C\}$

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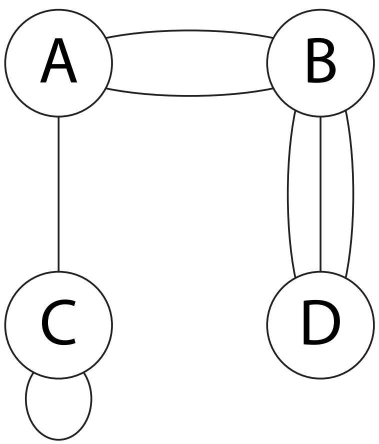

We will (almost) exclusively consider simple graphs in 374.

Why Graphs?

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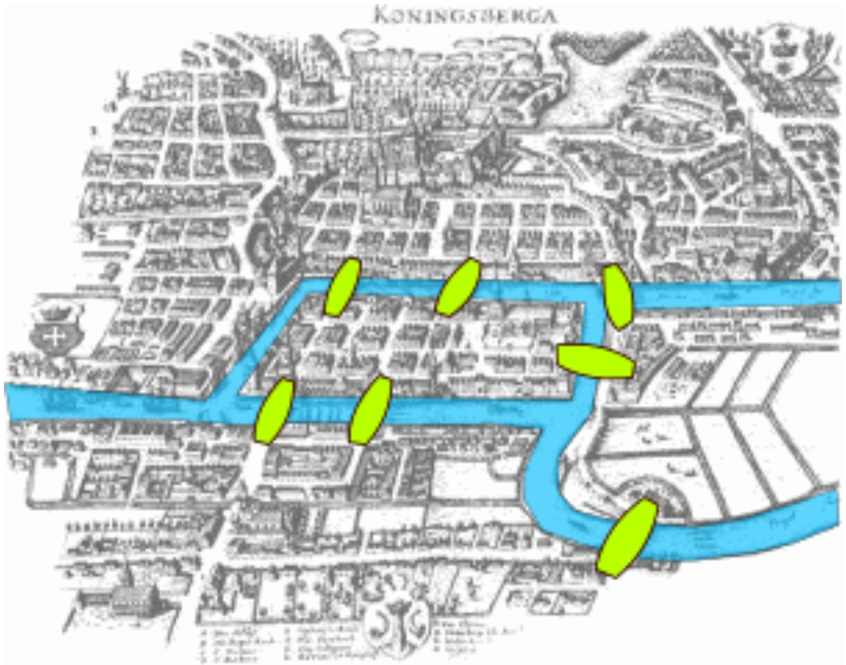
- *Simple* enough to admit nice problem statements, proofs, and algorithms.
- *Abstract* enough to allow us to phrase problems we care about in terms of a graph problem, filtering out unneeded details.

Graph Example: Bridges of Köningsberg

In the map below, can we find a route that crosses each bridge *exactly* once?



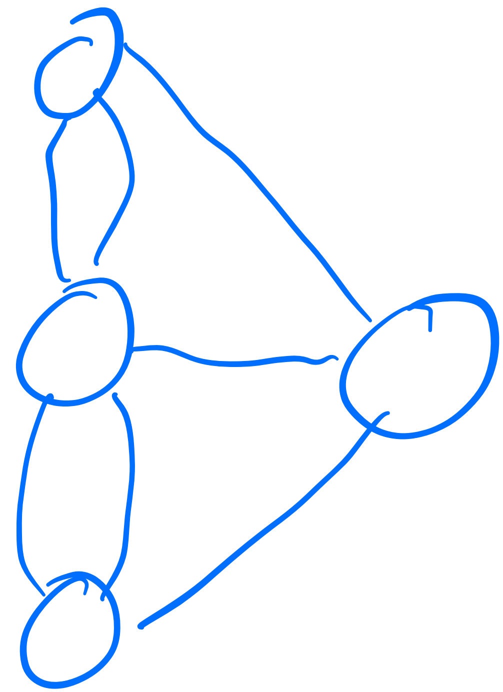


Image source: Wikipedia

Graph Example: Walking to Class

What's the fastest route to walk from Siebel to Lincoln Hall?

Image source: UIUC

$V = \{ \text{intersections} \}$

$E = \{ ij \mid i \text{ is directly connected to } j \}$

$w(e) =$ time it would take to walk the road corresponding to e

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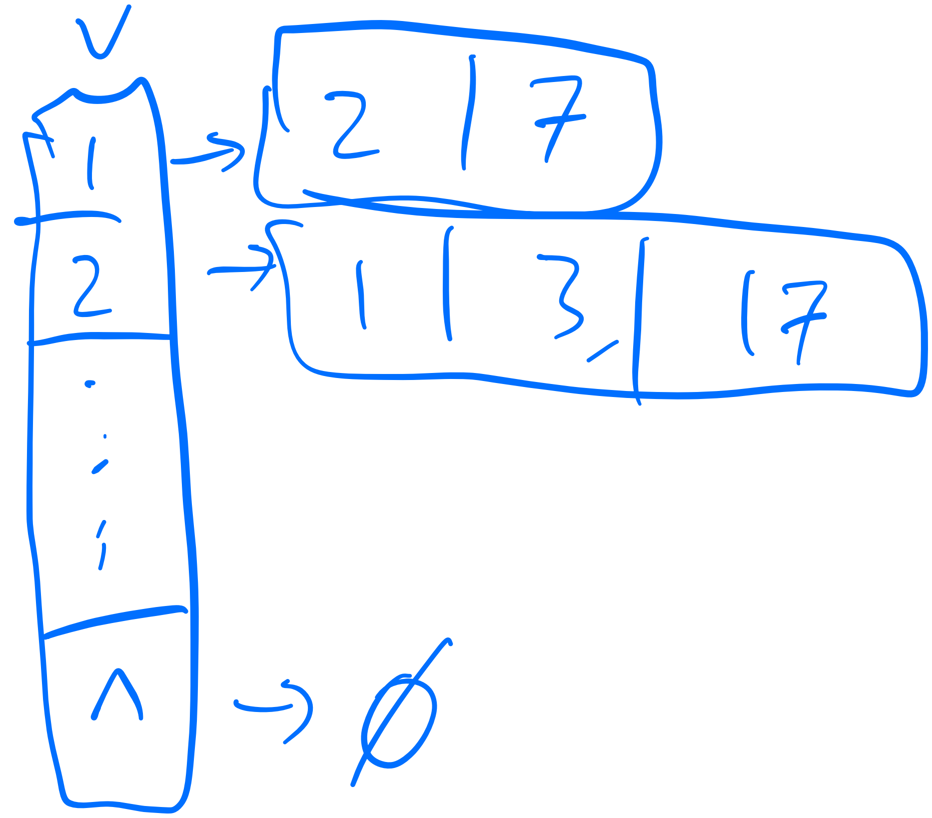
- Lecture: See common graph problems and efficient algorithms to solve them.
- Lab / Homework / Exams: Given a problem (not necessarily phrased in terms of graphs), reduce it to a problem on graphs and apply a known algorithm to solve it.
- "In the wild": Want to use someone else's (optimized, bug-free) implementation of a graph algorithm to solve your particular problem of interest.

Part II

Graph Data Structures

Adjacency List

For each vertex, store a list of all adjacent vertices.



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Efficiency for graph with ***n*** vertices and ***m*** edges?

- Space $O(n^2)$ $O(n+m)$
- Check adjacency $O(n)^*$ $O(\log n)$ if lists are sorted
  $O(\deg(v))$
- Iterate over edges
  $O(n+m)^*$ $O(m)$ if we maintain a separate list of edges

Adjacency Matrix

Store an $n \times n$ matrix of bits, where $A[i,j] = 1$ iff $\{i,j\} \in E$.

$$
\begin{array}{c|cccc|}
 & 1 & 2 & \cdots & n \\
\hline
1 & 0 & 1 & \cdots & 0 \\
2 & 1 & 0 & \cdots & 1 \\
\vdots & & \cdots & & \\
n & & & & \\
\hline
\end{array}
$$

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- Space $O(n^2)$
- Check adjacency $O(1)$
- Iterate over edges $O(n^2)$

Which Do I Use?

Whether an adjacency list or matrix is better will depend on the problem, as well as how you solve it.

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By default, we'll assume that input graphs are represented by adjacency lists, without either of the optimizations.

- If you want to change to a different representation, make sure your run time analysis accounts for that!

Part III

Connectivity in Undirected Graphs

Definitions

For a graph $G = (V, E)$:

- A path is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that for all $1 \leq i \leq k-1$, $\{v_i, v_{i+1}\} \in E$.

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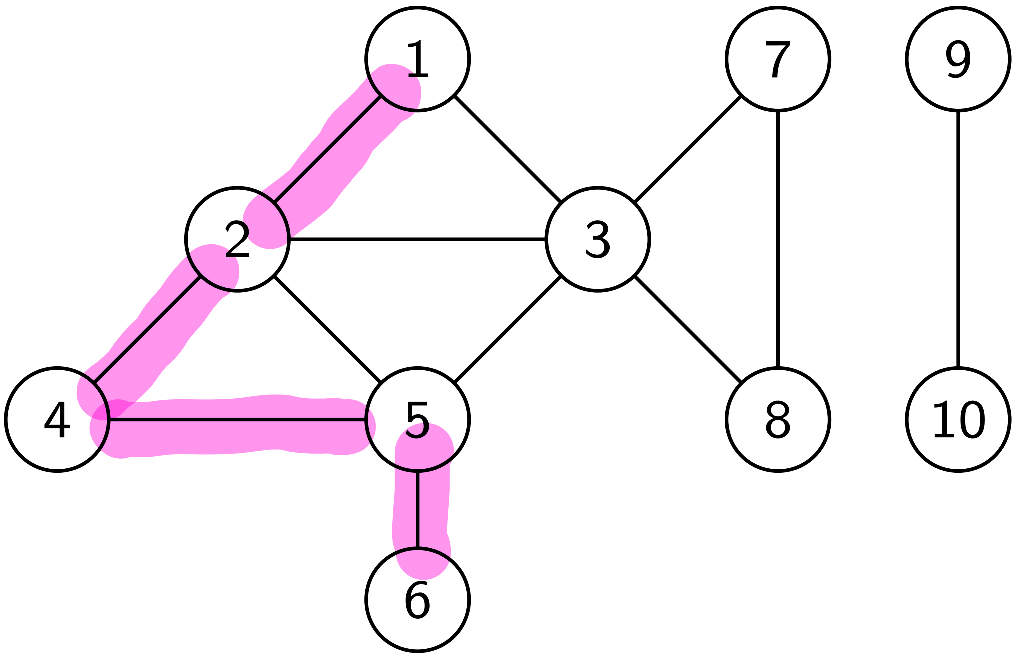
  - The length of a cycle is k (the number of edges).
  - The requirement that $k \geq 3$ ensures that a single edge does not count as a cycle.
- A vertex u is connected to v if there is a path with $v_1 = u$ and $v_k = v$.

Definitions Examples



$\mathbf{1, 2, 4, 5, 6}$ is a path.

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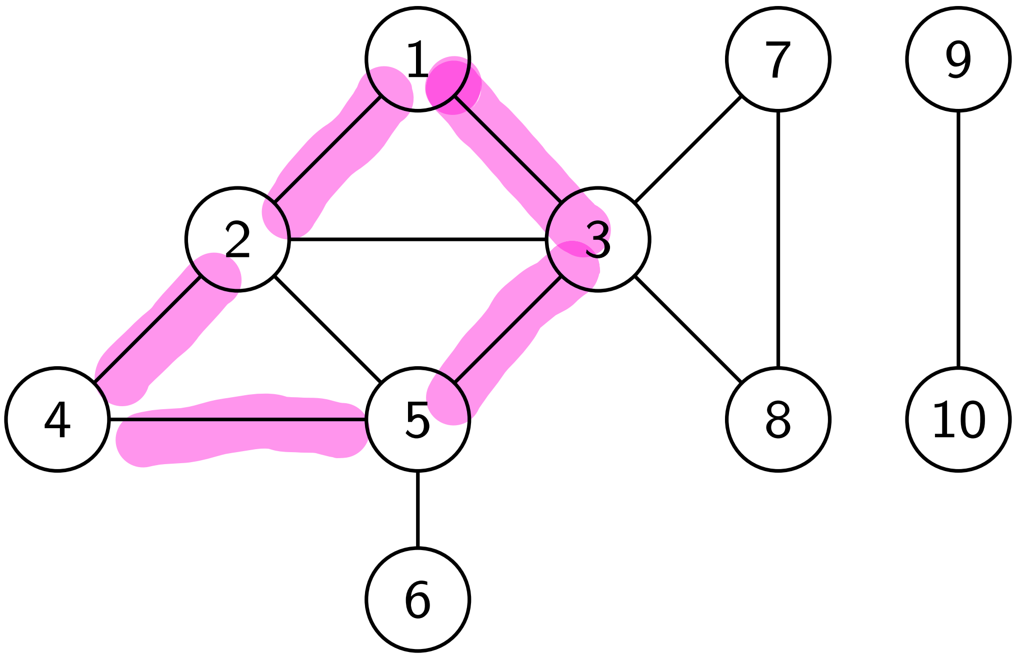

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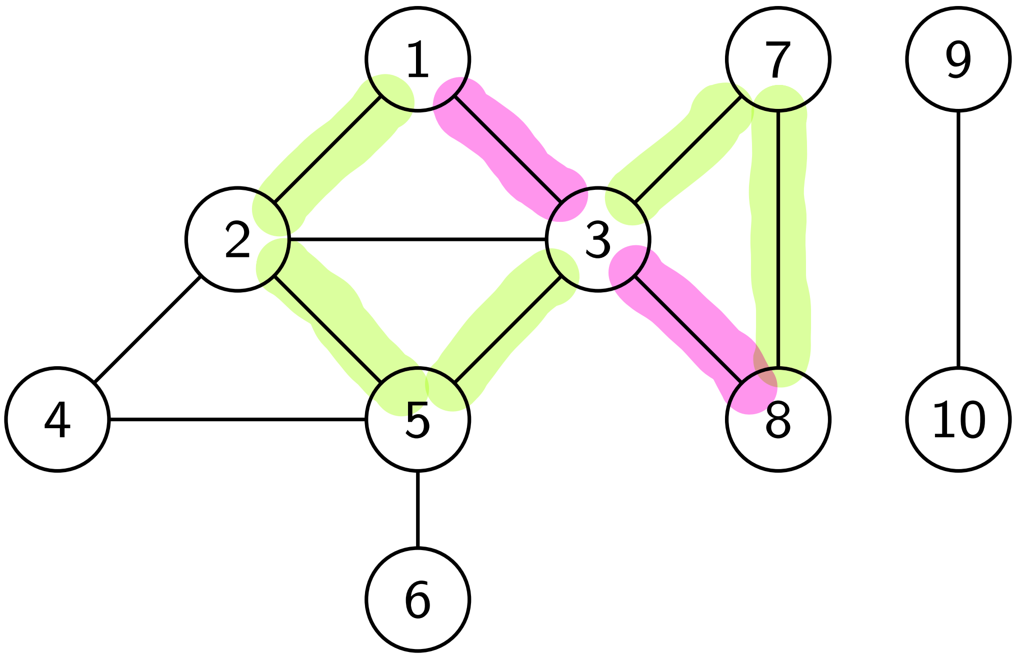

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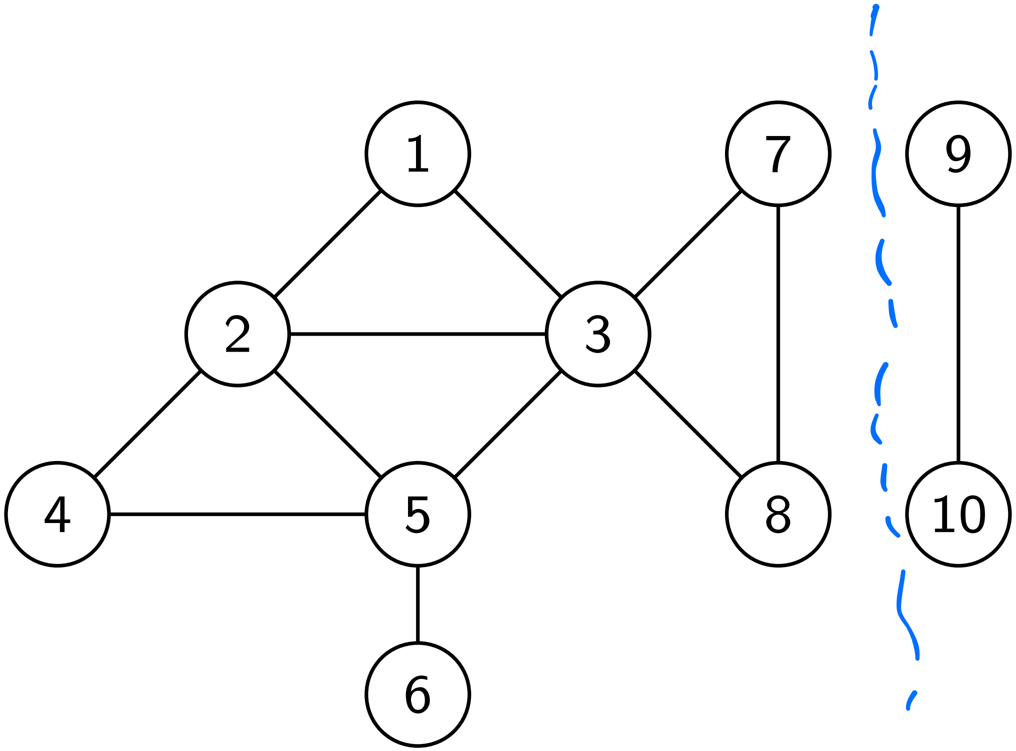

1, 2, 4, 5, 6 is a path.

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1 is *not* connected to **10**.

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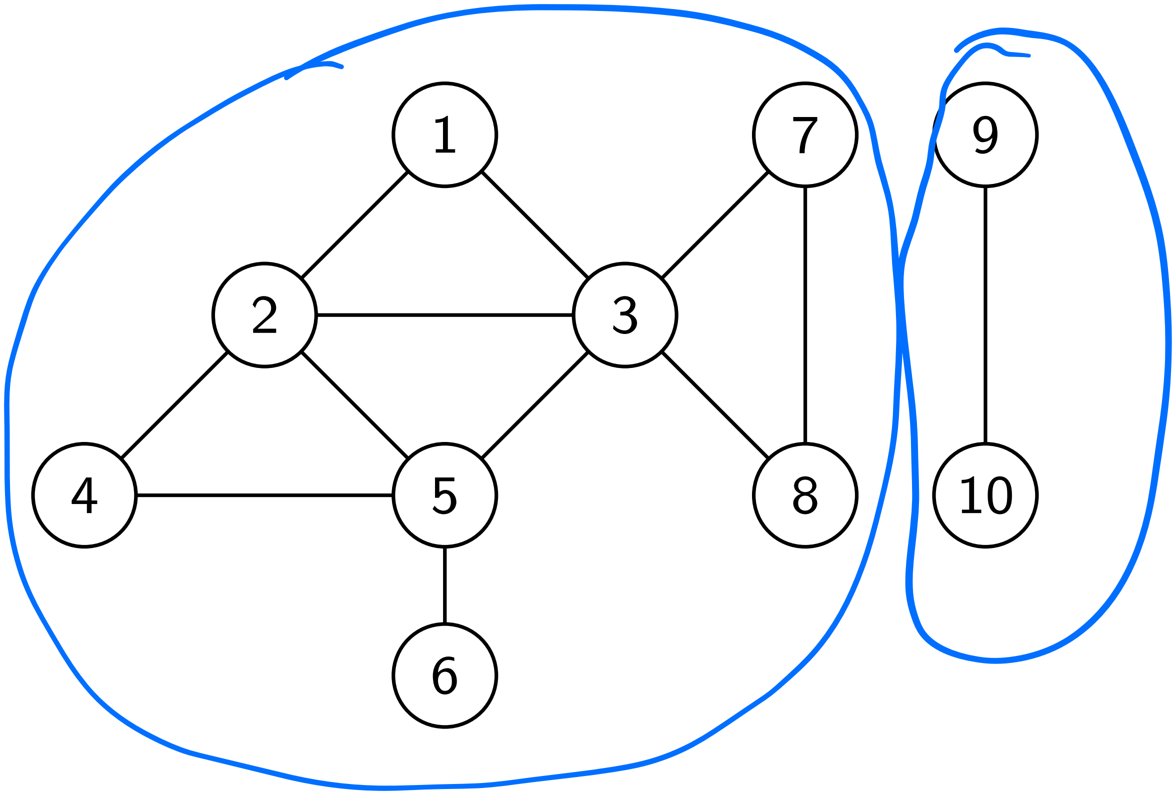
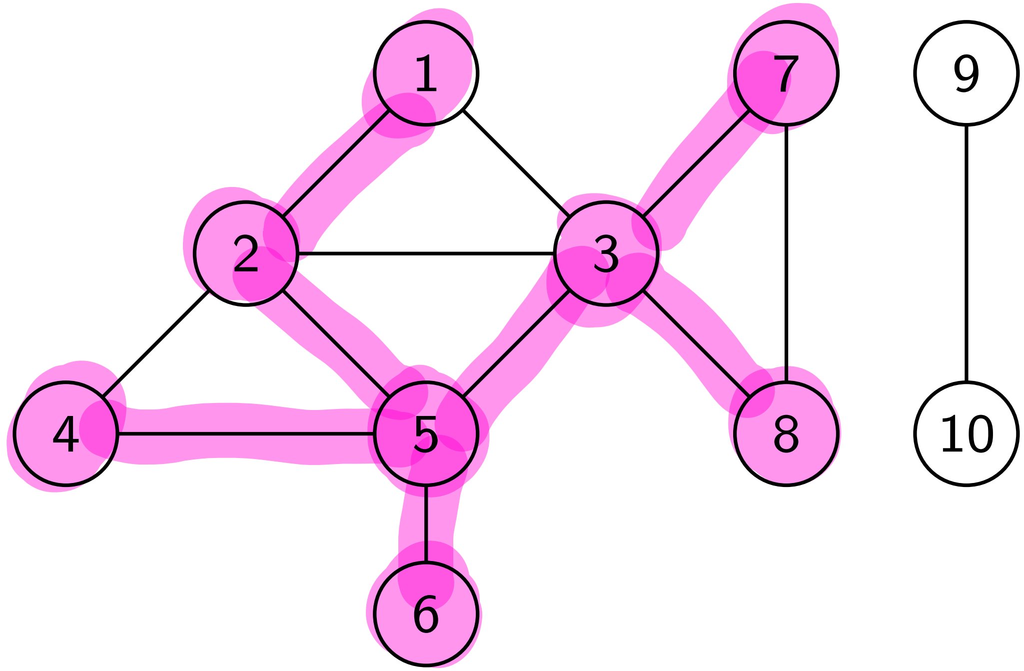


Algorithmic Connectivity

Let $G = (V, E)$ be an undirected graph. We want to solve the following problems:

1. Given nodes u and v, is u connected to v?
2. Given a node u, find the connected component containing u.
3. Find all connected components in G.

Intuitive Example



How would you find the connected component containing 6?

Naïve Search

Naïve(G, u):

  $S \leftarrow \{u\}$

  while (there is edge $\overset{vw}{\cancel{uv}}$ with $\overset{v}{\cancel{u}} \in S$, $\overset{w}{\cancel{v}} \notin S$) do

    $S \leftarrow S \cup \{\underset{w}{\cancel{v}}\}$

  Output S

Naïve Search

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Naïve(G,u):
    S ← {u}
    while (there is edge uv with u ∈ S, v ∉ S) do
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Correctness?

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$O(n+m)$ time to check

repeat at most n times

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Run time? $O(n \cdot (n+m)) \approx O(n \cdot m)$

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Can we do better?

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WFS(G,u):
    Initialize S = ∅
    Initialize ToExplore = {u}
    while ToExplore is non-empty
        Remove a node x from ToExplore
        if x is not in S
            Add x to S
            for each edge xy in Adj(x)
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Efficiency of WFS

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≤ 2m times

Only considers each edge twice

$O(n+m)$ time

Breadth and Depth First Search

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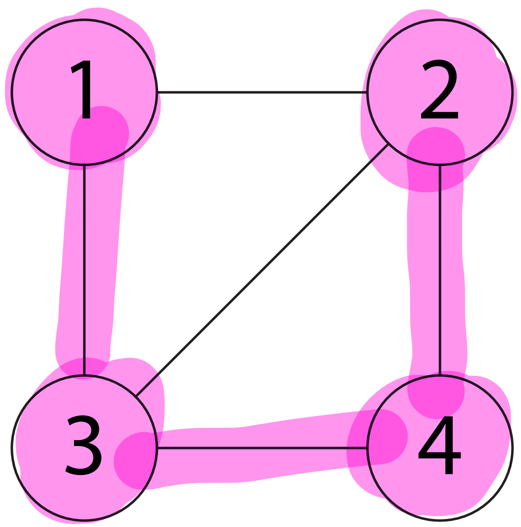
- Stack (LIFO): Depth-first search
- Queue (FIFO): Breadth-first search

BFS and DFS Examples

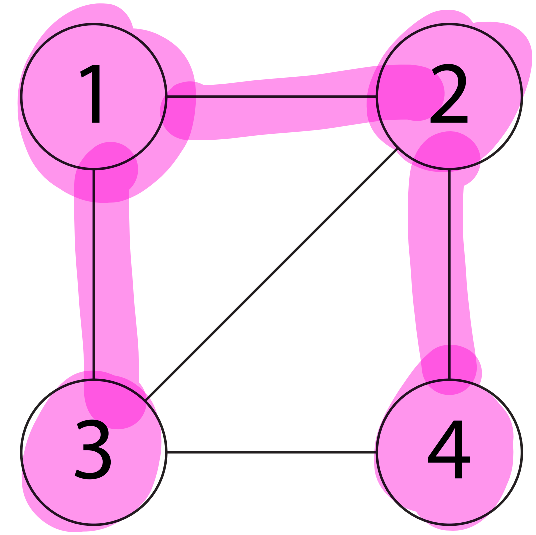
DFS



$\texttt{ToExplore} =$

$\{ \quad \}$

BFS



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The tree for **DFS** has some more subtle properties that we'll explore in the next lecture.