

Backtracking and Memoization

Lecture 12

Mar 4, 2025

Recursion

Reduction:

Reduce one problem to another

Recursion

A special case of reduction

- 1 reduce problem to a *smaller* instance of *itself*
 - 2 self-reduction
-
- 1 Problem instance of size n is reduced to one or more instances of size $n - 1$ or less.
 - 2 For termination, problem instances of small size are solved by some other method as **base cases**.

Recursion in Algorithm Design

- 1 **Tail Recursion**: problem reduced to a *single* recursive call after some work. Easy to convert algorithm into iterative or greedy algorithms. Examples: Interval scheduling, MST algorithms, etc.
- 2 **Divide and Conquer**: Problem reduced to multiple **independent** sub-problems that are solved separately. Conquer step puts together solution for bigger problem.
Examples: merge sort, quick sort, multiplication, selection
- 3 **Backtracking**: Refinement of brute force search. Build solution incrementally by invoking recursion to try all possibilities for the decision in each step.
- 4 **Dynamic Programming**: problem reduced to multiple (typically) *dependent or overlapping* sub-problems. Use **memoization** to avoid recomputation of common solutions leading to *iterative bottom-up* algorithm.

Subproblems in Recursion

- Suppose $foo()$ is a *recursive* program/algorithm for a problem.
- Given an instance I , $foo(I)$ generates potentially many “smaller” problems.
- If $foo(I')$ is one of the calls during the execution of $foo(I)$ we say I' is a subproblem of I .
- Recursive execution can be viewed as a tree.
- The *same* subproblem I' may occur more than once in the recursion tree.
- Number of *distinct* subproblems will be an important measure.

Subproblems in Recursion

```
foo(l):  
  do stuff  
  x = foo(l1)  
  do stuff  
  y = foo(l2)  
  Output blah
```

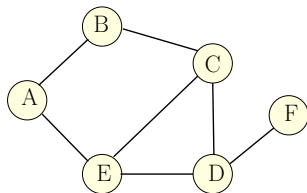
Part I

Brute Force Search, Recursion and Backtracking

Maximum Independent Set in a Graph

Definition

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is an **independent set** (also called a stable set) if for there are no edges between nodes in S . That is, if $u, v \in S$ then $(u, v) \notin E$.

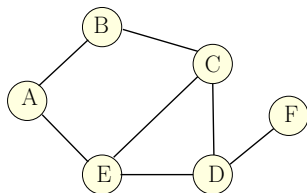


Some independent sets in graph above: $\{D\}$, $\{A, C\}$, $\{B, E, F\}$

Maximum Independent Set Problem

Input Graph $G = (V, E)$

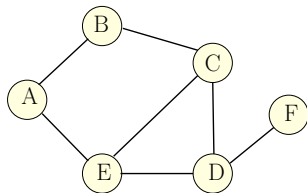
Goal Find maximum sized independent set in G



Maximum Weight Independent Set Problem

Input Graph $G = (V, E)$, weights $w(v) \geq 0$ for $v \in V$

Goal Find maximum weight independent set in G



Maximum Weight Independent Set Problem

- 1 No one knows an *efficient* (polynomial time) algorithm for this problem
- 2 Problem is **NP-Complete** and it is *believed* that there is no polynomial time algorithm

Brute-force algorithm:

Try all subsets of vertices.

Brute-force enumeration

Algorithm to find the size of the maximum weight independent set.

```
MaxIndSet( $G = (V, E)$ ):
```

```
   $max = 0$ 
```

```
  for each subset  $S \subseteq V$  do
```

```
    check if  $S$  is an independent set
```

```
    if  $S$  is an independent set and  $w(S) > max$  then
```

```
       $max = w(S)$ 
```

```
  Output  $max$ 
```

Brute-force enumeration

Algorithm to find the size of the maximum weight independent set.

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```

```
       $max = w(S)$ 
```

```
  Output  $max$ 
```

Running time: suppose G has n vertices and m edges

- 1 2^n subsets of V
- 2 checking each subset S takes $O(m)$ time
- 3 total time is $O(m2^n)$

A Recursive Algorithm

Let $V = \{v_1, v_2, \dots, v_n\}$.

For a vertex u let $N(u)$ be its neighbors.

A Recursive Algorithm

Let $V = \{v_1, v_2, \dots, v_n\}$.

For a vertex u let $N(u)$ be its neighbors.

Observation

v_1 : vertex in the graph.

\mathcal{S} : set of independent sets that contain v_1

\mathcal{S}' : set of independent sets that do not contain v_1

Find max weight independent set from \mathcal{S} and \mathcal{S}' . Take the better of the two. Each case allows us to “reduce” the size of the problem.

A Recursive Algorithm

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$G_1 = G - v_1$ obtained by removing v_1 and incident edges from G

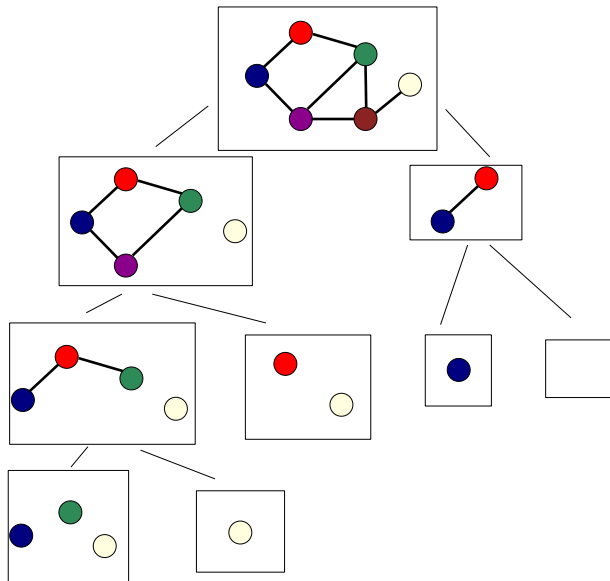
$G_2 = G - v_1 - N(v_1)$ obtained by removing $N(v_1) \cup v_1$ from G

$$MIS(G) = \max\{MIS(G_1), MIS(G_2) + w(v_1)\}$$

A Recursive Algorithm

```
RecursiveMIS( $G$ ):  
  if  $G$  is empty then Output 0  
   $a = \text{RecursiveMIS}(G - v_1)$   
   $b = w(v_1) + \text{RecursiveMIS}(G - v_1 - N(v_n))$   
  Output  $\max(a, b)$ 
```


Example



Recursive Algorithms

..for Maximum Independent Set

Running time:

$$T(n) = T(n - 1) + T(n - 1 - \text{deg}(v_1)) + O(1 + \text{deg}(v_1))$$

where $\text{deg}(v_1)$ is the degree of v_1 . $T(0) = T(1) = 1$ is base case.

Worst case is when $\text{deg}(v_1) = 0$ when the recurrence becomes

$$T(n) = 2T(n - 1) + O(1)$$

Solution to this is $T(n) = O(2^n)$.

Backtrack Search via Recursion

- 1 Recursive algorithm generates a tree of computation where each node is a smaller problem (subproblem)
- 2 Simple recursive algorithm computes/explores the whole tree blindly in some order.
- 3 Backtrack search is a way to explore the tree intelligently to prune the search space
 - 1 Some subproblems may be so simple that we can stop the recursive algorithm and solve it directly by some other method
 - 2 Memoization to avoid recomputing same problem
 - 3 Stop the recursion at a subproblem if it is clear that there is no need to explore further.
 - 4 Leads to a number of heuristics that are widely used in practice although the worst case running time may still be exponential.

Sequences

Definition

Sequence: an ordered list a_1, a_2, \dots, a_n . **Length** of a sequence is number of elements in the list.

Definition

a_{i_1}, \dots, a_{i_k} is a **subsequence** of a_1, \dots, a_n if
 $1 \leq i_1 < i_2 < \dots < i_k \leq n$.

Definition

A sequence is **increasing** if $a_1 < a_2 < \dots < a_n$. It is **non-decreasing** if $a_1 \leq a_2 \leq \dots \leq a_n$. Similarly **decreasing** and **non-increasing**.

Sequences

Example...

Example

- 1 Sequence: 6, 3, 5, 2, 7, 8, 1, 9
- 2 Subsequence of above sequence: 5, 2, 1
- 3 Increasing sequence: 3, 5, 9, 17, 54
- 4 Decreasing sequence: 34, 21, 7, 5, 1
- 5 Increasing subsequence of the first sequence: 2, 7, 9.

Longest Increasing Subsequence Problem

Input A sequence of numbers a_1, a_2, \dots, a_n

Goal Find an **increasing subsequence** $a_{i_1}, a_{i_2}, \dots, a_{i_k}$ of maximum length

Longest Increasing Subsequence Problem

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Example

- 1 Sequence: 6, 3, 5, 2, 7, 8, 1
- 2 Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
- 3 Longest increasing subsequence: 3, 5, 7, 8

Naïve Enumeration

Assume a_1, a_2, \dots, a_n is contained in an array A

```
algLISNaive( $A[1..n]$ ):  
   $max = 0$   
  for each subsequence  $B$  of  $A$  do  
    if  $B$  is increasing and  $|B| > max$  then  
       $max = |B|$   
  
  Output  $max$ 
```


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Running time:

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```

Running time: $O(n2^n)$.

2^n subsequences of a sequence of length n and $O(n)$ time to check if a given sequence is increasing.

Recursive Approach: Take 1

LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

LIS($A[1..n]$):

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- 1 Case 1: max without $A[n]$ which is $\text{LIS}(A[1..(n-1)])$
- 2 Case 2: max among sequences that contain $A[n]$ in which case recursion is

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- 2 Case 2: max among sequences that contain $A[n]$ in which case recursion is not so clear.

Observation

For second case we want to find a subsequence in $A[1..(n-1)]$ that is restricted to numbers less than $A[n]$. This suggests that a more general problem is $\text{LIS_smaller}(A[1..n], x)$ which gives the longest increasing subsequence in A where each number in the sequence is less than x .

Recursive Approach

LIS_smaller($A[1..n]$, x): length of longest increasing subsequence in $A[1..n]$ with all numbers in subsequence less than x

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LIS_smaller($A[1..n]$, x): length of longest increasing subsequence in $A[1..n]$ with all numbers in subsequence less than x

```
LIS_smaller( $A[1..n]$ ,  $x$ ):  
  if ( $n = 0$ ) then return 0  
   $m = \text{LIS\_smaller}(A[1..(n - 1)], x)$   
  if ( $A[n] < x$ ) then  
     $m = \max(m, 1 + \text{LIS\_smaller}(A[1..(n - 1)], A[n]))$   
  Output  $m$ 
```

```
LIS( $A[1..n]$ ):  
  return LIS_smaller( $A[1..n]$ ,  $\infty$ )
```


Example

Sequence: $A[1..7] = 6, 3, 5, 2, 7, 8, 1$

Part II

Recursion and Memoization

Fibonacci Numbers

Fibonacci numbers defined by recurrence:

$$F(n) = F(n - 1) + F(n - 2) \text{ and } F(0) = 0, F(1) = 1.$$

These numbers have many interesting and amazing properties.
A journal *The Fibonacci Quarterly*!

- 1 $F(n) = (\phi^n - (1 - \phi)^n) / \sqrt{5}$ where ϕ is the golden ratio $(1 + \sqrt{5})/2 \simeq 1.618$.
- 2 $\lim_{n \rightarrow \infty} F(n + 1) / F(n) = \phi$

How many bits?

Consider the n th Fibonacci number $F(n)$. Writing the number $F(n)$ in base 2 requires

- $\Theta(n^2)$ bits.
- $\Theta(n)$ bits.
- $\Theta(\log n)$ bits.
- $\Theta(\log \log n)$ bits.

Recursive Algorithm for Fibonacci Numbers

Question: Given n , compute $F(n)$.

```
Fib( $n$ ):  
  if ( $n = 0$ )  
    return 0  
  else if ( $n = 1$ )  
    return 1  
  else  
    return Fib( $n - 1$ ) + Fib( $n - 2$ )
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Running time? Let $T(n)$ be the number of additions in $\text{Fib}(n)$.

$$T(n) = T(n - 1) + T(n - 2) + 1 \text{ and } T(0) = T(1) = 0$$

Roughly same as $F(n)$

$$T(n) = \Theta(\phi^n)$$

The number of additions is exponential in n . Can we do better?

Iterative algorithm for Fibonacci numbers

```
FibIter( $n$ ):  
  if ( $n = 0$ ) then  
    return 0  
  if ( $n = 1$ ) then  
    return 1  
   $F[0] = 0$   
   $F[1] = 1$   
  for  $i = 2$  to  $n$  do  
     $F[i] = F[i - 1] + F[i - 2]$   
  return  $F[n]$ 
```

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What is the running time of the algorithm?

Iterative algorithm for Fibonacci numbers

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    return 0  
  if ( $n = 1$ ) then  
    return 1  
   $F[0] = 0$   
   $F[1] = 1$   
  for  $i = 2$  to  $n$  do  
     $F[i] = F[i - 1] + F[i - 2]$   
  return  $F[n]$ 
```

What is the running time of the algorithm? $O(n)$ additions.

What is the difference?

- 1 Recursive algorithm is computing the same numbers repeatedly.
- 2 Iterative algorithm is storing computed values and building bottom up the final value.

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Dynamic Programming:

Finding a recursion that can be *effectively/efficiently* memoized.

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.

Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

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```
Fib( $n$ ):  
  if ( $n = 0$ )  
    return 0  
  if ( $n = 1$ )  
    return 1  
  if (Fib( $n$ ) was previously computed)  
    return stored value of Fib( $n$ )  
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How do we keep track of previously computed values?

Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

```
Fib(n):  
  if (n = 0)  
    return 0  
  if (n = 1)  
    return 1  
  if (Fib(n) was previously computed)  
    return stored value of Fib(n)  
  else  
    return Fib(n - 1) + Fib(n - 2)
```

How do we keep track of previously computed values?

Two methods: **explicitly** and **implicitly** (via data structure)

Automatic explicit memoization

Initialize array $M[n + 1]$ such that $M[i] = -1$ for $i = 0, \dots, n$.

Automatic explicit memoization

Initialize array $M[n + 1]$ such that $M[i] = -1$ for $i = 0, \dots, n$.

```
Fib( $n$ ):  
  if ( $n = 0$ )  
    return 0  
  if ( $n = 1$ )  
    return 1  
  if ( $M[n] \neq -1$ ) (*  $M[n]$  has stored value of Fib( $n$ ) *)  
    return  $M[n]$   
   $M[n] \leftarrow$  Fib( $n - 1$ ) + Fib( $n - 2$ )  
  return  $M[n]$ 
```

To allocate memory need to know upfront the number of distinct subproblems for a given input size n

Automatic implicit memoization

Initialize a (dynamic) dictionary data structure D to empty

```
Fib( $n$ ):  
  if ( $n = 0$ )  
    return 0  
  if ( $n = 1$ )  
    return 1  
  if ( $n$  is already in  $D$ )  
    return value stored with  $n$  in  $D$   
     $val \leftarrow \mathbf{Fib}(n - 1) + \mathbf{Fib}(n - 2)$   
  Store ( $n, val$ ) in  $D$   
  return  $val$ 
```

Explicit vs Implicit Memoization

- 1 Explicit memoization or iterative algorithm preferred if one can analyze problem ahead of time. Allows for efficient memory allocation and access.
- 2 Implicit and automatic memoization used when problem structure or algorithm is either not well understood or in fact unknown to the underlying system.
 - 1 Need to pay overhead of data-structure.
 - 2 Functional languages such as LISP automatically do memoization, usually via hashing based dictionaries.
 - 3 Python has library for automatic memoization of functions.

Automatic memoization in Python

```
#!/usr/bin/python3
import functools
import time

def fib_reg(n):
    if (n == 0):
        return 1
    if (n == 1):
        return 1
    return fib_reg(n-1) + fib_reg(n-2)

@functools.cache
def fib_mem(n):
    if (n == 0):
        return 1
    if (n == 1):
        return 1
    return fib_mem(n-1) + fib_mem(n-2)

start = time.time()
print ("fib(200) = ", fib_mem(200))
end = time.time()
print ("Time to compute fib(200) with memoization: ", end-start)

for i in range(35,39):
    start = time.time()
    print ("fib(%d) = %d" % (i, fib_reg(i)))
    end = time.time()
    print ("Time to compute fib(%d) without memoization: " % i, end-start)
```

Automatic memoization in Python

```
chekuri@chandras-mini Code % python3 fib-memoization.py
fib(200) = 453973694165307953197296969697410619233826
Time to compute fib(200) with memoization: 0.0001361370086669922
fib(35) = 14930352
Time to compute fib(35) without memoization: 1.9982337951660156
fib(36) = 24157817
Time to compute fib(36) without memoization: 3.2325918674468994
fib(37) = 39088169
Time to compute fib(37) without memoization: 5.178303003311157
fib(38) = 63245986
Time to compute fib(38) without memoization: 8.395797967910767
chekuri@chandras-mini Code %
```


Implementation in C and overflow

```
#include<stdio.h>

int fib_reg(int n)
{
    if (n < 0)
        return -1;

    if (n == 0)
        return 1;
    if (n == 1)
        return 1;

    return fib_reg(n-1) + fib_reg(n-2);
}

int fib_iter(int n)
{
    if (n < 0)
        return -1;

    if (n == 0 || n == 1)
        return 1;

    int prev = 1, old = 1, val;

    for (int i = 2; i <= n; i++) {
        val = prev + old;
        old = prev;
        prev = val;
    }

    return val;
}

int main()
{
    printf("Fib(50) = %d\n", fib_iter(50));
}
```

Implementation in C and overflow

```
chekuri@chandras-mini Code % ./fib.out  
Fib(50) = -1109825406  
chekuri@chandras-mini Code %
```

How many distinct subproblems?

```
binom(t, b)    // computes  $\binom{t}{b}$   
  if t = 0 then return 0  
  if b = t or b = 0 then return 1  
  return binom(t - 1, b - 1) + binom(t - 1, b).
```

How many *distinct subproblems* does **binom**(*n*, $\lfloor n/2 \rfloor$) generate its recursive execution?

- $\Theta(1)$
- $\Theta(n)$
- $\Theta(n \log n)$
- $\Theta(n^2)$
- $\Theta\left(\binom{n}{\lfloor n/2 \rfloor}\right)$

Running time of memoized binom?

```
D: Initially an empty dictionary.  
binomM(t, b) // computes  $\binom{t}{b}$   
  if b = t then return 1  
  if b = 0 then return 0  
  if D[t, b] is defined then return D[t, b]  
  D[t, b]  $\leftarrow$  binomM(t - 1, b - 1) + binomM(t - 1, b).  
  return D[t, b]
```

Assuming that every arithmetic operation takes $O(1)$ time, What is the running time of **binomM**(*n*, $\lfloor n/2 \rfloor$)?

- $\Theta(1)$
- $\Theta(n)$
- $\Theta(n^2)$
- $\Theta(n^3)$
- $\Theta\left(\binom{n}{\lfloor n/2 \rfloor}\right)$

Back to Fibonacci Numbers

Is the iterative algorithm a *polynomial* time algorithm? Does it take $O(n)$ time?

Back to Fibonacci Numbers

Is the iterative algorithm a *polynomial* time algorithm? Does it take $O(n)$ time?

- 1 input is a number n and hence input size in binary representation is $\Theta(\log n)$
- 2 output is $F(n)$ and output size is $\Theta(n)$. Why?
- 3 Hence output size is exponential in input size so no polynomial time algorithm possible!
- 4 Running time of iterative algorithm: $\Theta(n)$ additions but number sizes are $O(n)$ bits long! Hence total time is $O(n^2)$, in fact $\Theta(n^2)$. Why?

Back to Fibonacci Numbers

Saving space. Do we need an array of n numbers? Not really.

```
FibIter( $n$ ):  
  if ( $n = 0$ ) then  
    return 0  
  if ( $n = 1$ ) then  
    return 1  
   $prev2 = 0$   
   $prev1 = 1$   
  for  $i = 2$  to  $n$  do  
     $temp = prev1 + prev2$   
     $prev2 = prev1$   
     $prev1 = temp$   
  
  return  $prev1$ 
```