NFAs continued, Closure Properties of Regular Languages

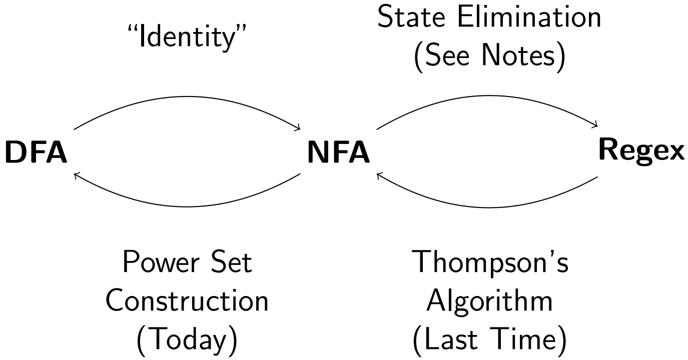
Lecture 5

February 04, 2025

Part I

Equivalence of DFAs, NFAs, and Regular Expressions

Roadmap

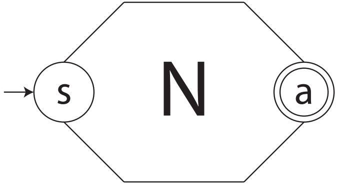


"Identity"
State Elimination
(See Notes)
DFA
NFA
Regex
Power Set
Construction
(Today)
Thompson's
Algorithm
(Last Time)

Last Time on CS374

Theorem

For every regular expression r, *there is a normal form NFA* N *such that* $L(N) = L(r)$.



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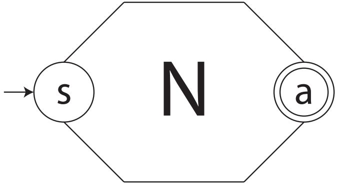

Last time, dealt with base cases ($\emptyset$, ϵ, single character).

Thompson's Algorithm: Union

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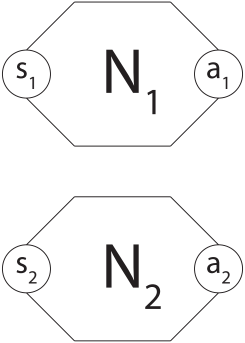
$r = r_1 + r_2$



Thompson's Algorithm: Concatenation

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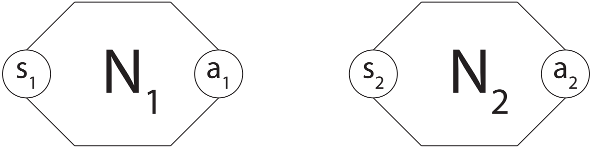

Thompson's Algorithm: Kleene Star

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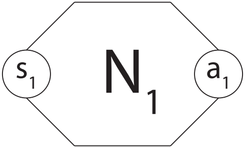

Thompson's Algorithm: Kleene Star (Attempt 2)

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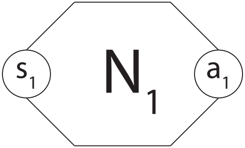

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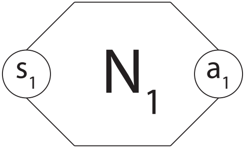

Thompson's Algorithm: Example

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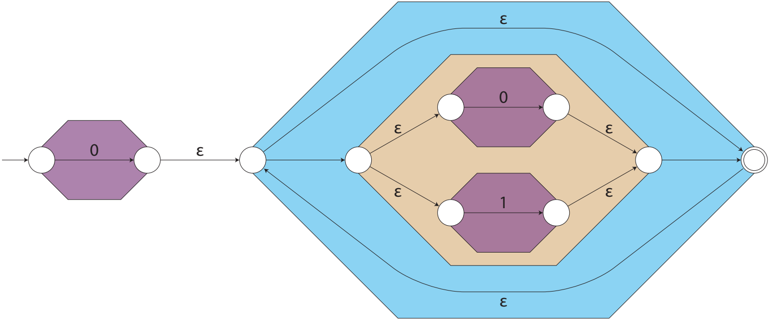

Powerset Construction: Intuition

An NFA accepts a string w if it is *possible* to reach an accepting state when reading the string w.

How would we work out by hand if an accepting path is possible?

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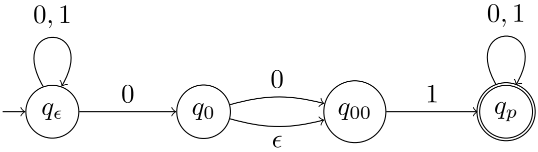

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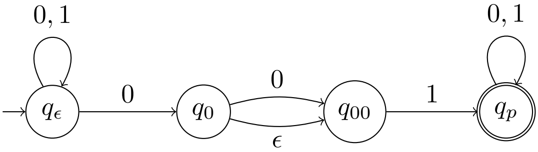

Is there an accepting path for **1001**?
What about for **1100**?

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When we could be at an accepting state.

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- $Q_M = \mathcal{P}(Q_N)$
- $s_M = \epsilon\text{reach}(s_N)$
- $\delta_M(q_M, a) = \bigcup_{q_N \in q_M} \bigcup_{r_N \in \delta_N(q_N, a)} \epsilon\text{reach}(r_N)$
- $A_M = \{q_M | q_M \cap A_N \neq \emptyset\}$

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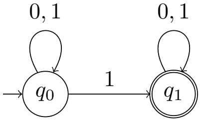
Powerset Construction: Example



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- So this lemma says N accepts w if and only if M does!

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Note: Base case ($w = \epsilon$) remains the same.

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$|x| < |w|$, so by the IH $\delta_N^*(s_N, x) = \delta_M^*(s_M, x)$!

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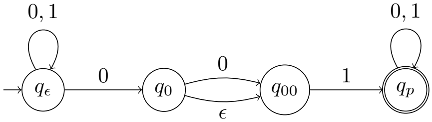
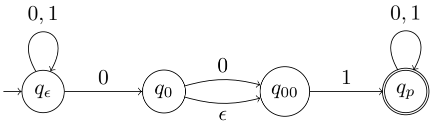
- This is sometimes unavoidable: some languages *require* DFAs that are exponentially bigger than their smallest NFA.
- However, for many NFAs, applying the powerset construction will result in states that can't be reached from the start state, and so can be ignored.

When applying the powerset construction by hand, we can simply add states "as needed" to avoid having too many.

Incremental Construction: Example



Incremental Construction: Tabular



q'	in A'?	$\bigcup_{q \in q'} \delta(q, 0)$	$\delta'(q', 0)$	$\bigcup_{q \in q'} \delta(q, 1)$	$\delta'(q', 1)$

Part II

Closure Properties of Regular Languages

Properties of Regular Languages

Regular languages have three equivalent characterizations:

- Languages defined by regular expressions
- Languages accepted by DFAs
- Languages accepted by NFAs

We can use any of these three to prove properties about regular languages.

- The complement of a regular language is regular: take a DFA and invert the accepting states.
- Boolean combinations of regular languages (union, intersection, difference, ...) are regular: take a DFA for each and apply the product construction
- Concatenation, Kleene star via NFAs or regular expressions
- ...

Generic Closure Property

In general, closure properties will look something like the following:

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Let L be a regular language and L' be [some modification of L]. Then L' is also a regular language.

How would we go about proving a statement like this?

- To show that L' is regular, we just have to argue that it has a regular expression/DFA/ NFA.
- We know L is regular, so it has a regular expression/DFA/NFA.
- Try to modify or use the regular expression/DFA/NFA for L in order to build what we want for L'

Example 1: PREFIX

Let L be a language over Σ.

Definition

$\text{PREFIX}(L) = \{w \mid \exists x \in \Sigma^* \text{ s.t. } wx \in L\}$

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Intuitive idea:

- Want to know if it's possible to extend our string to get something in L.

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- Want to know if it's possible to extend our string to get something in L.
- Phrased in terms of a DFA for L: is it still possible for us to reach an accepting state by reading additional characters?

Example 1: PREFIX

Theorem

If L is regular then PREFIX(L) is regular.

Let $M = (Q, \delta, s, A)$ be a DFA for L. Construct a DFA $M' = (Q', \delta', s', A')$ for PREFIX(L) as:

- $Q' =$
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- $Q' = Q$
- $\delta'(q', a) = \delta(q', a)$
- $s' = s$
- $A' = \{q \in Q \mid \text{in } M, q \text{ can reach a state in } A\}$

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Intuitive idea:

- Want to know if it's possible to have read something *before* our string to get something in L.
- Phrased in terms of a DFA for L: could we run for some time before reading w and end up in an accepting state?
- How do we know what state to go to before starting to read w? Guess!

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- $Q' = Q \cup \{s_{new}\}$
- $\delta'(q', a) =$
$$\begin{cases} \{\delta(q', a)\} & q' \in Q, a \in \Sigma \\ \{q \in Q \mid s \text{ can reach } q \text{ in } M\} & q' = s_{new}, a = \epsilon \\ \emptyset & \text{otherwise} \end{cases}$$
- $s' = s_{new}$
- $A' = A$

Example 3: SUFFIX, Again

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Alternative proof: inductively show that we can convert a regular expression r for L to a regular expression r' for SUFFIX(L).

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$r = a$ $r' =$	$r = (r_1)^*$ $r' =$

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$r = a$ $r' = \epsilon + a$	$r = (r_1)^*$ $r' = r'_1 (r_1)^*$

Exercises

Given language L, let $\text{MID}(L) = \{w \mid \exists x, y \in \Sigma^*, xwy \in L\}$.

Theorem

If L is regular then $MID(L)$ is regular.

Given L let $L^R = \{w^R \mid w \in L\}$ be the reverses of all strings in L

Theorem

If L is regular then L^R is regular.