NFAs continued, Closure Properties of Regular Languages

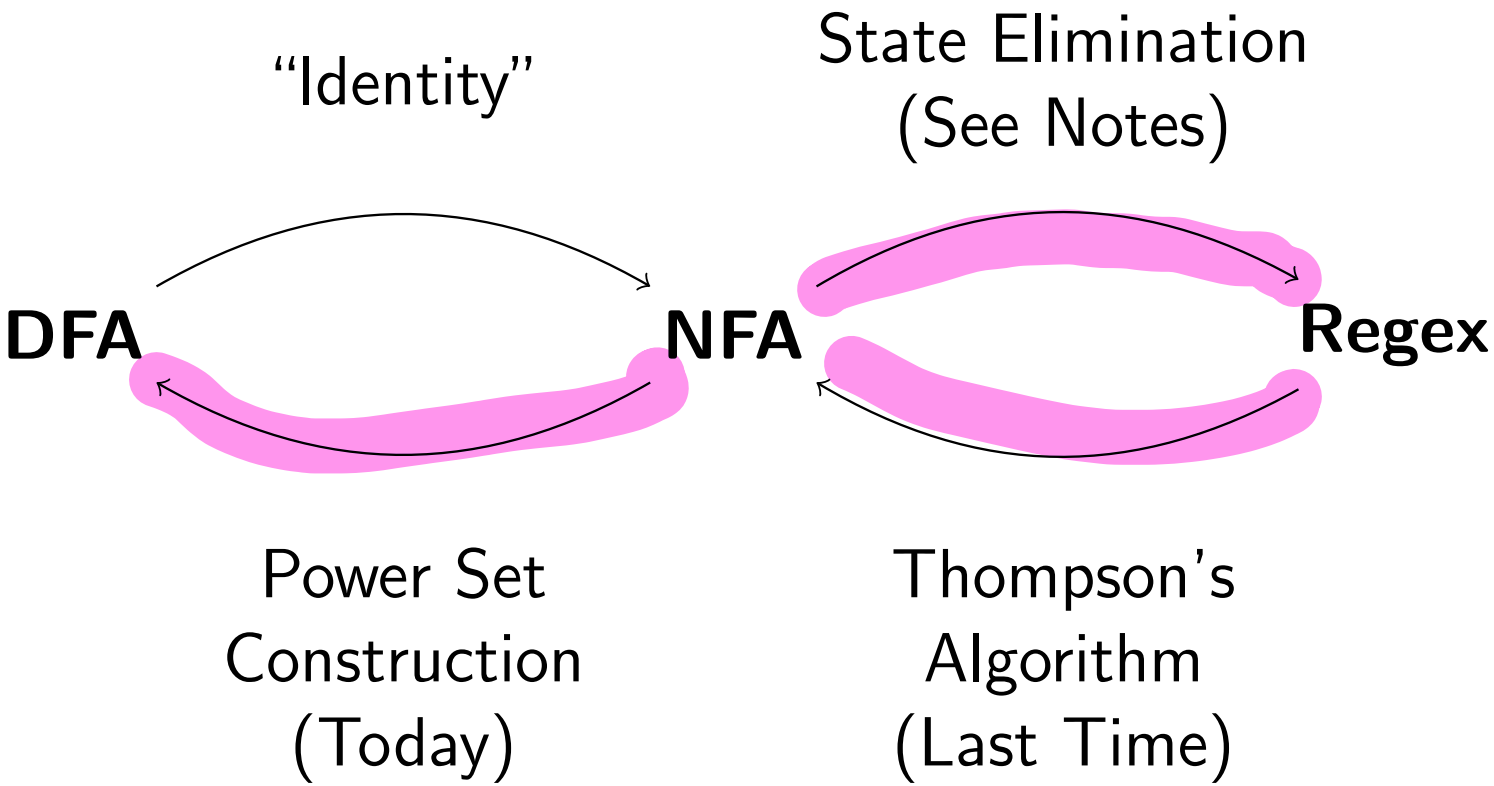
Lecture 5

January 31, 2025

Part I

Equivalence of DFAs, NFAs, and Regular Expressions

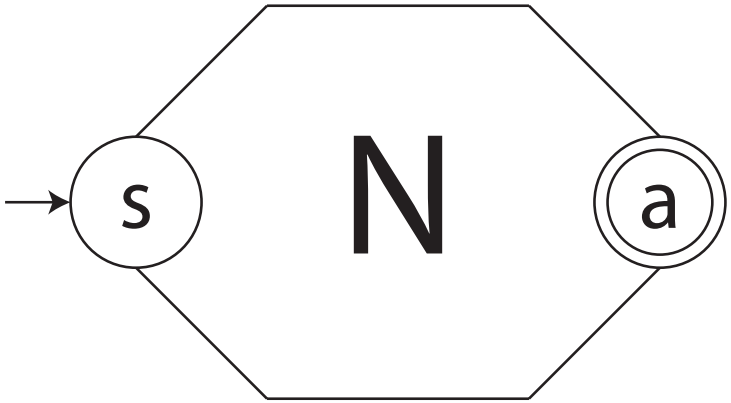
Roadmap


"Identity"
State Elimination
(See Notes)
DFA
NFA
Regex
Power Set
Construction
(Today)
Thompson's
Algorithm
(Last Time)

Last Time on CS374

Theorem

For every regular expression r, there is a normal form NFA N such that $L(N) = L(r)$.



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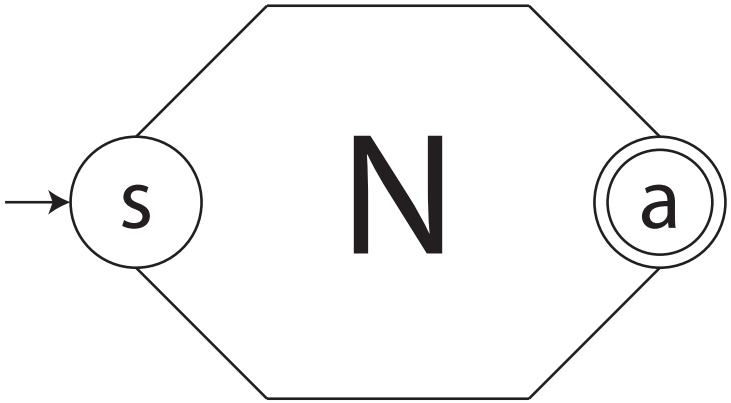

Last time, dealt with base cases ($\emptyset$, ϵ, single character).

Thompson's Algorithm: Union

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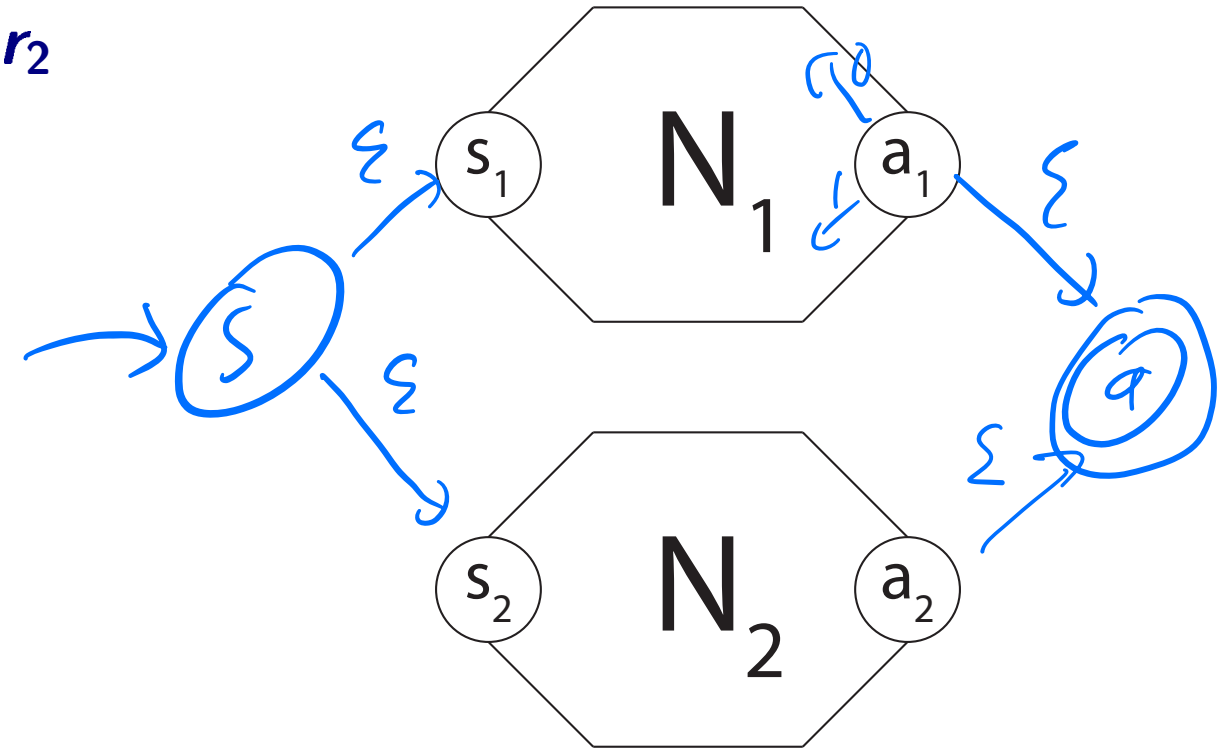
$r = r_1 + r_2$



Thompson's Algorithm: Concatenation

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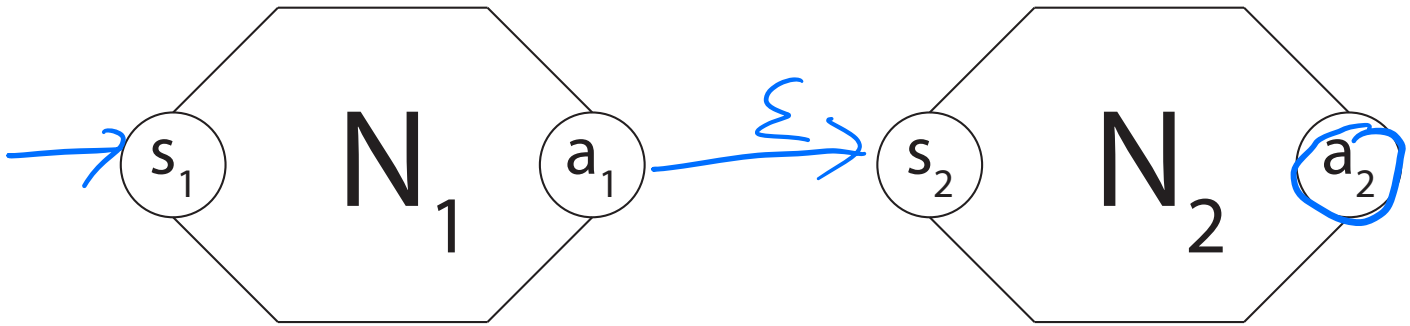

Thompson's Algorithm: Kleene Star

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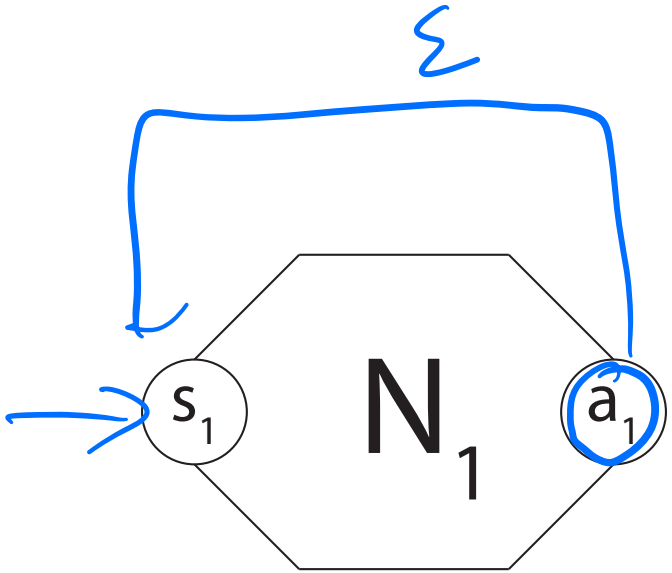

doesn't accept ε

Thompson's Algorithm: Kleene Star (Attempt 2)

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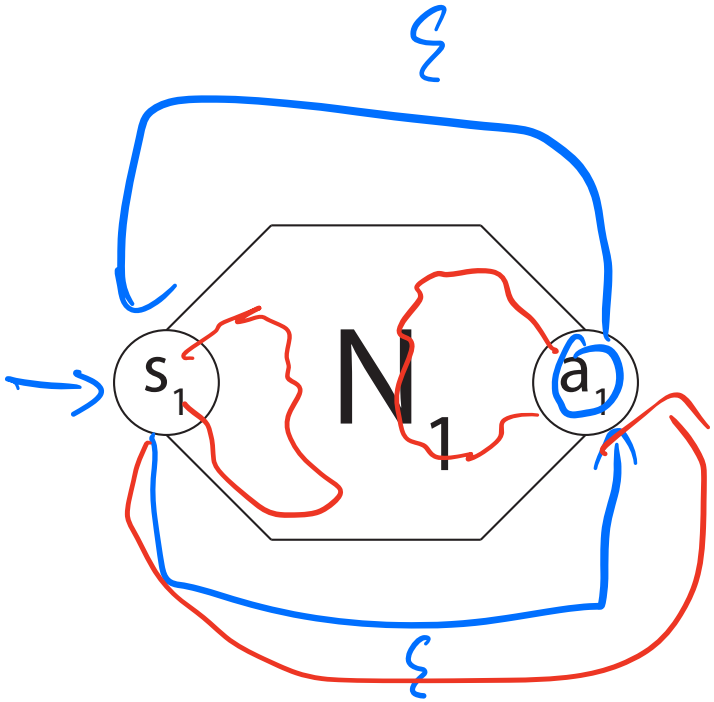

ε-transitions can be followed in the middle of a string

Thompson's Algorithm: Kleene Star (Attempt 3)

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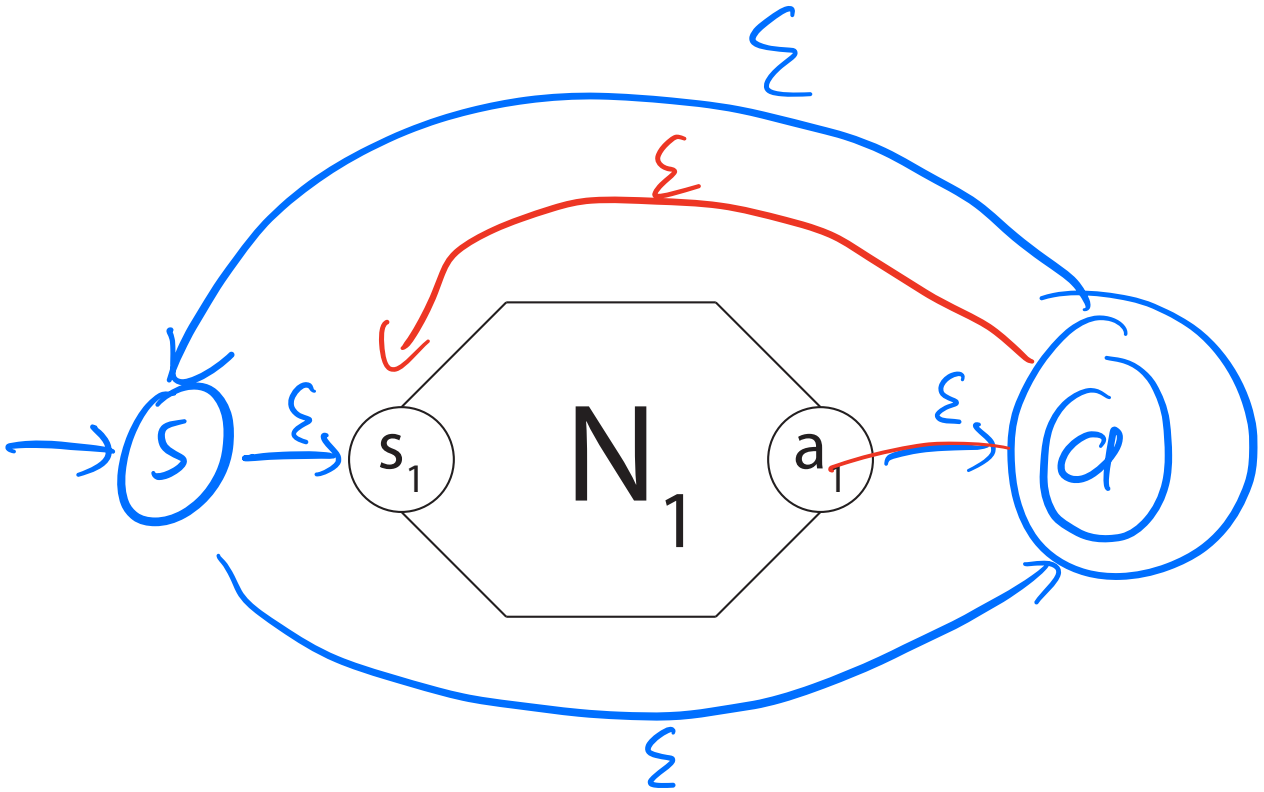

Thompson's Algorithm: Example

$r = 0(0 + 1)^*$

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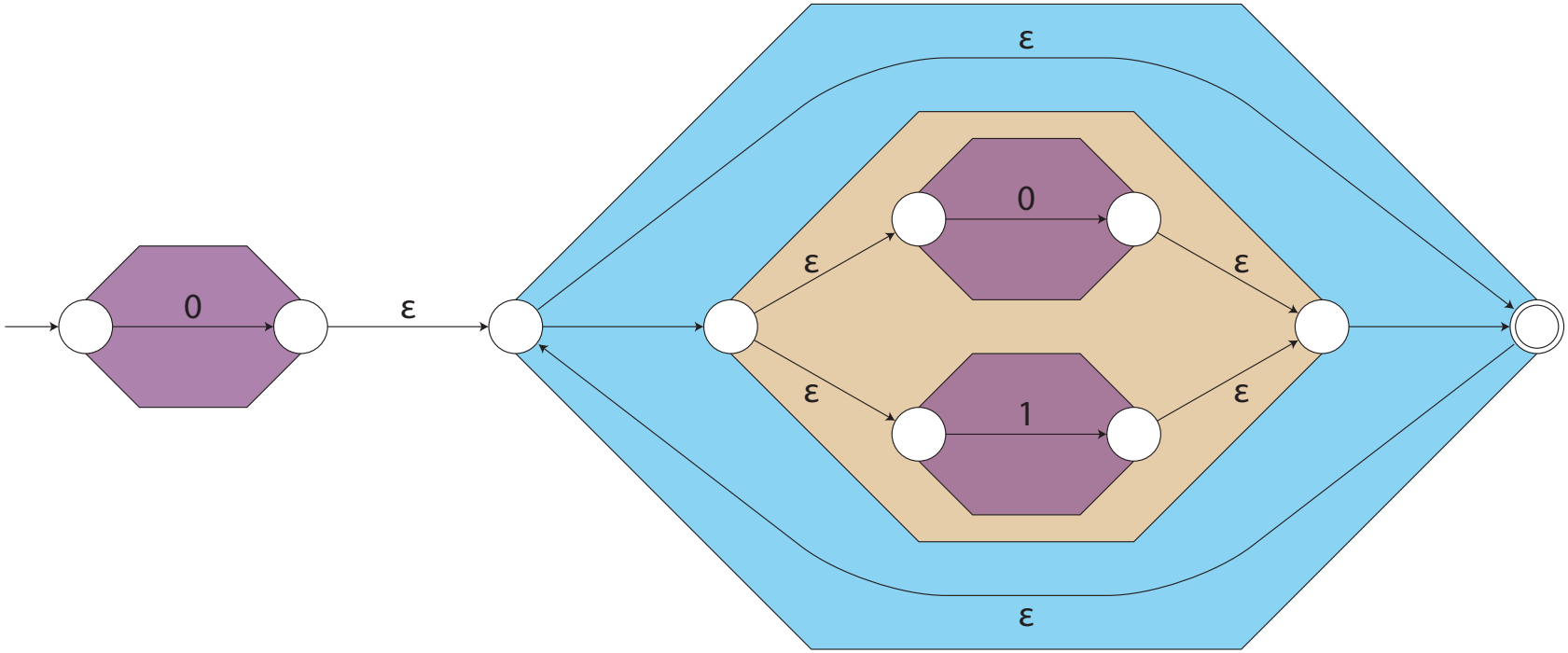

Powerset Construction: Intuition

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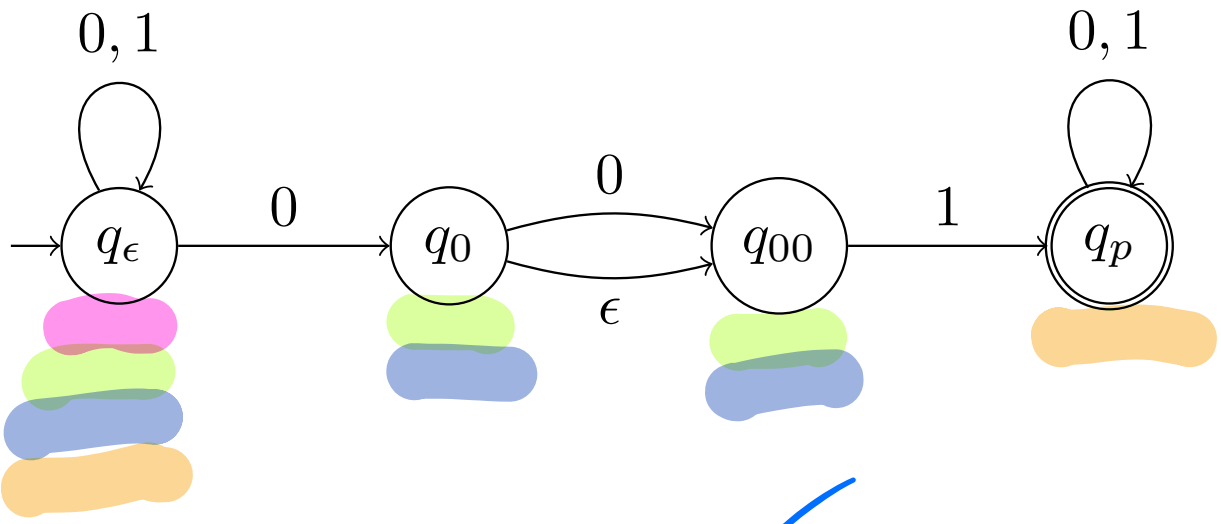

Is there an accepting path for **1001**? ✓

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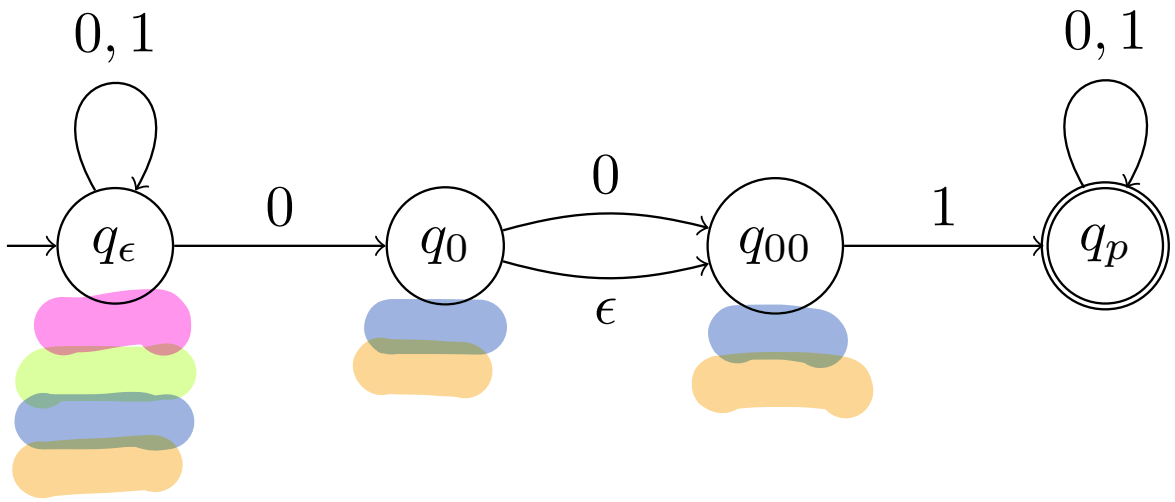

Is there an accepting path for **1001**?
What about for **1100**?

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For $N = (Q_N, \Sigma, s_N, \delta_N, A_N)$, create $M = (Q_M, \Sigma, \delta_M, s_M, A_M)$:

- $Q_M = P(Q_N)$ "power set"
- $s_M = \{ reach(s_N) \}$
- $\delta_M(q_M, a) = \bigcup_{q_N \in q_M} \bigcup_{r_N \in \delta_N(q_N, a)} \{ reach(r_N) \}$
- $A_M = \{ q_M \mid q_M \cap A_N \neq \emptyset \}$

Intuition: track "set of possible states".

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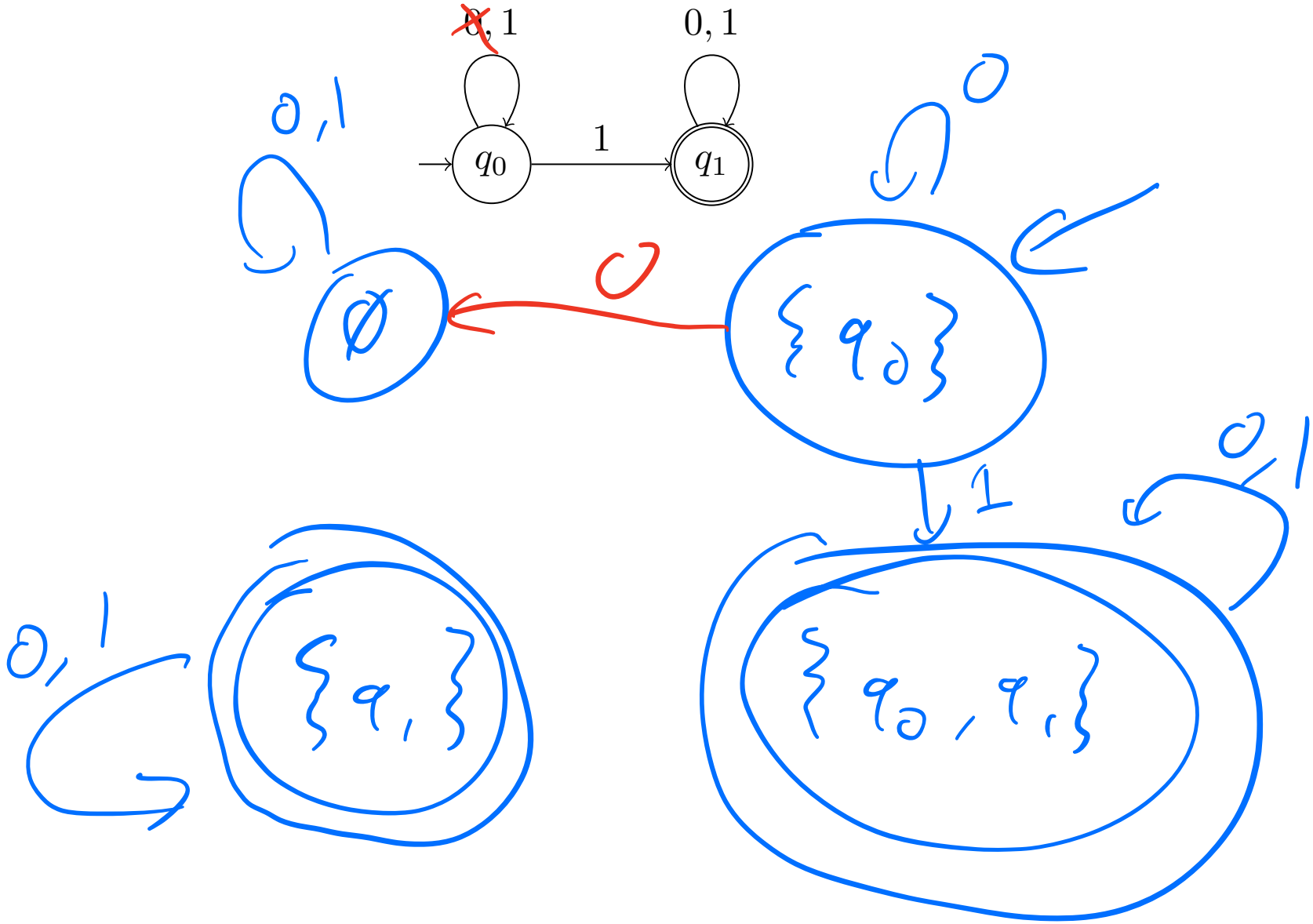

Proof of Correctness: Lemma Statement

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- So this lemma says N accepts w if and only if M does!

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Let N be an NFA and M be the DFA obtained by applying the powerset construction to N. For all w, $\delta^*_N(s_N, w) = \delta^*_M(s_M, w)$.

We will prove this by induction on the length of w.

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Note: Base case ($w = \epsilon$) remains the same.

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$|x| < |w|$, so by the IH $\delta_N^*(s_N, x) = \delta_M^*(s_M, x)$!

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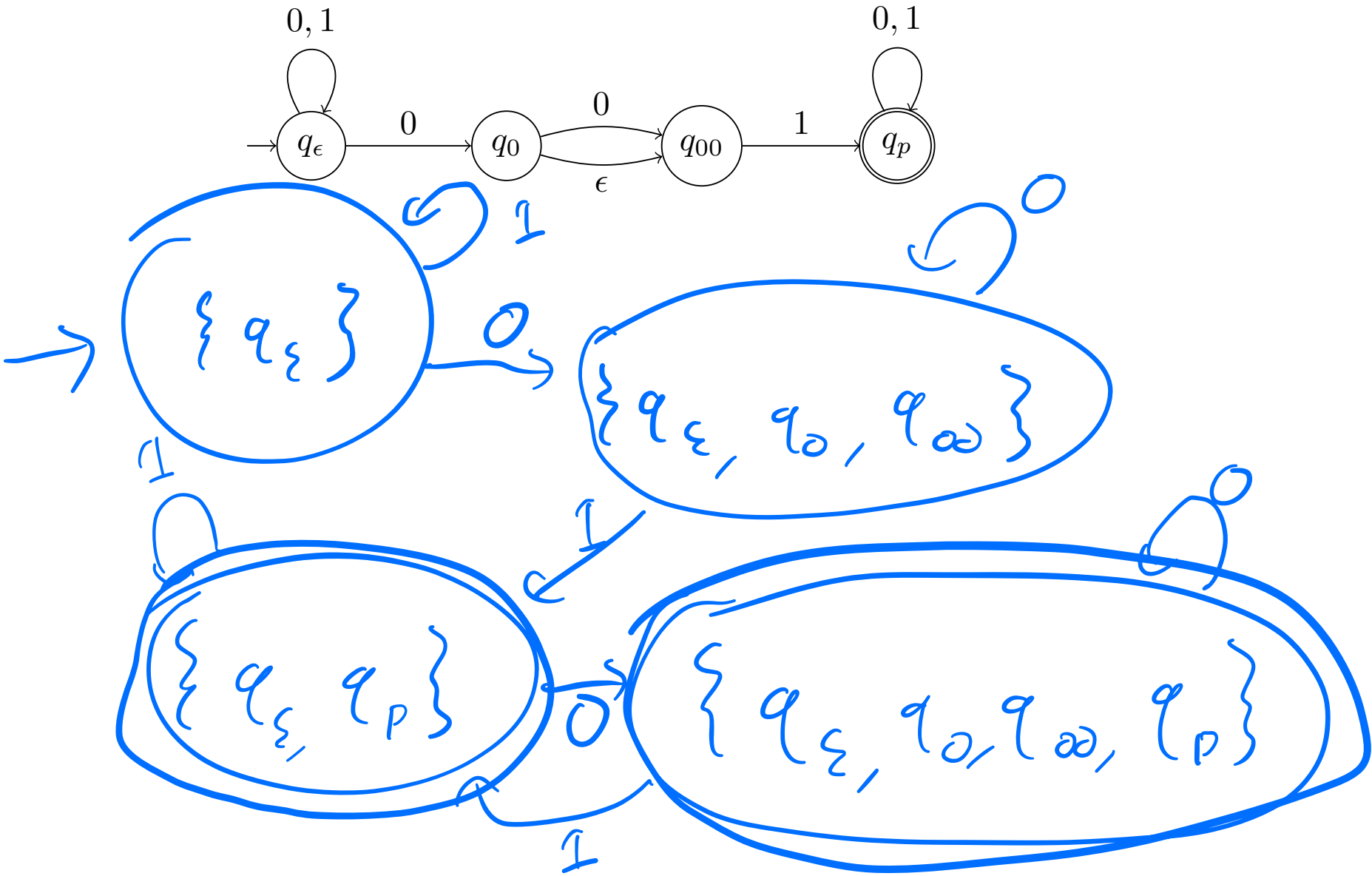
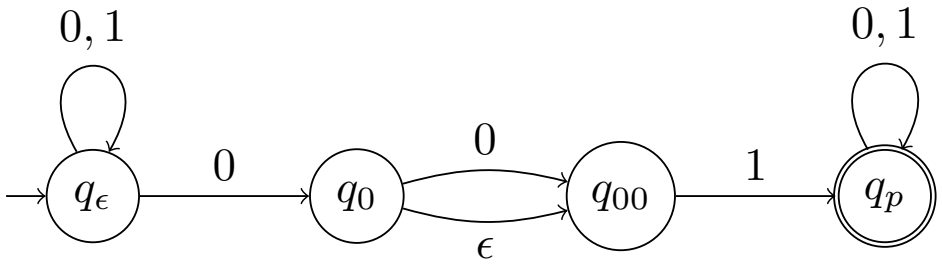
- However, for many NFAs, applying the powerset construction will result in states that can't be reached from the start state, and so can be ignored.

When applying the powerset construction by hand, we can simply add states "as needed" to avoid having too many.

Incremental Construction: Example



Incremental Construction: Tabular



q'	in A'?	$\bigcup_{q \in q'} \delta(q, 0)$	$\delta'(q', 0)$	$\bigcup_{q \in q'} \delta(q, 1)$	$\delta'(q', 1)$
$\{q_\epsilon\}$	N	$\{q_\epsilon, q_0\}$	$\{q_\epsilon, q_0, q_{00}\}$	...	

Part II

Closure Properties of Regular Languages

Properties of Regular Languages

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Rest of today: generalize how we prove such "closure properties"

Generic Closure Property

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- We know L is regular, so has a regular expression / DFA / NFA.
- Try to modify or use the regular expression / DFA / NFA for L in order to build what we want for L'!

Example 1: PREFIX

Let L be a language over Σ.

Definition

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Intuitive idea:

- Want to know if it's possible to extend our string to get something in L.
- Phrased in terms of a DFA for L: is it still possible for us to reach an accepting state by reading additional characters?

Example 1: PREFIX

Theorem

If L is regular then $PREFIX(L)$ is regular.

Let $M = (Q, \delta, s, A)$ be a DFA for L. Construct a DFA $M' = (Q', \delta', s', A')$ for $PREFIX(L)$ as:

- $Q' =$
- $\delta'(q', a) =$
- $s' =$
- $A' =$

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$\text{SUFFIX}(L) = \{w \mid \exists x \in \Sigma^* \text{ s.t. } xw \in L\}$

Theorem

If L is regular then $\text{SUFFIX}(L)$ is regular.

Intuitive idea:

- Want to know if it's possible to have read something *before* our string to get something in L.

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- How do we know what state to go to before starting to read w? Guess!

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Theorem

If L is regular then $PREFIX(L)$ is regular.

Let $M = (Q, \delta, s, A)$ be a DFA for L. Construct an NFA $N = (Q', \delta', s', A')$ for $SUFFIX(L)$ as:

- $Q' =$
- $\delta'(q', a) =$
- $s' =$
- $A' =$

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Alternative proof: inductively show that we can convert a regular expression r for L to a regular expression r' for SUFFIX(L).

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$r = \emptyset$ $r' =$	$r = r_1 + r_2$ $r' =$
$r = \epsilon$ $r' =$	$r = r_1 r_2$ $r' =$
$r = a$ $r' =$	$r = (r_1)^*$ $r' =$

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$r = a$ $r' = \epsilon + a$	$r = (r_1)^*$ $r' = r'_1 (r_1)^*$