Give context-free grammars for each of the following languages.

1. $\{\mathtt{0}^{2n}\mathtt{1}^n \mid n \geq 0\}$

   Solution: $S \to \varepsilon \mid \mathtt{00}S\mathtt{1}$ ■

2. $\{\mathtt{0}^m\mathtt{1}^n \mid m \neq 2n\}$

   [Hint: If $m \neq 2n$, then either $m < 2n$ or $m > 2n$.]

   Solution: To simplify notation, let $\Delta(w) = \#(\mathtt{0}, w) - 2\#(\mathtt{1}, w)$. Our solution follows the following logic. Let w be an arbitrary string in this language.

   - Because $\Delta(w) \neq 0$, then either $\Delta(w) > 0$ or $\Delta(w) < 0$.
   - If $\Delta(w) > 0$, then $w = \mathtt{0}^i z$ for some integer $i > 0$ and some suffix z with $\Delta(z) = 0$.
   - If $\Delta(w) < 0$, then $w = x\mathtt{1}^j$ for some integer $j > 0$ and some prefix x with either $\Delta(x) = 0$ or $\Delta(x) = 1$.
   - Substrings with $\Delta = 0$ is generated by the previous grammar; we need only a small tweak to generate substrings with $\Delta = 1$.

   Here is one way to encode this case analysis as a CFG. The nonterminals M and L generate all strings where the number of 0s is More or Less than twice the number of 1s, respectively. The last nonterminal generates strings with $\Delta = 0$ or $\Delta = 1$.

   $$\begin{aligned} S &\to M \mid L & \{\mathtt{0}^m\mathtt{1}^n \mid m \neq 2n\} \\ M &\to \mathtt{0}M \mid \mathtt{0}E & \{\mathtt{0}^m\mathtt{1}^n \mid m > 2n\} \\ L &\to L\mathtt{1} \mid E\mathtt{1} & \{\mathtt{0}^m\mathtt{1}^n \mid m < 2n\} \\ E &\to \varepsilon \mid \mathtt{0} \mid \mathtt{00}E\mathtt{1} & \{\mathtt{0}^m\mathtt{1}^n \mid m = 2n \text{ or } 2n+1\} \end{aligned}$$

   Here is a different correct solution using the same logic. We either identify a non-empty prefix of 0s or a non-empty prefix of 1s, so that the rest of the string as "balanced" as possible. We also generate strings with $\Delta = 1$ using a separate non-terminal.

   $$\begin{aligned} S &\to AE \mid EB \mid FB & \{\mathtt{0}^m\mathtt{1}^n \mid m \neq 2n\} \\ A &\to \mathtt{0} \mid \mathtt{0}A & \mathtt{0}^+ = \{\mathtt{0}^i \mid i \geq 1\} \\ B &\to \mathtt{1} \mid \mathtt{1}B & \mathtt{1}^+ = \{\mathtt{1}^j \mid j \geq 1\} \\ E &\to \varepsilon \mid \mathtt{00}E\mathtt{1} & \{\mathtt{0}^m\mathtt{1}^n \mid m = 2n\} \\ F &\to \mathtt{0}E & \{\mathtt{0}^m\mathtt{1}^n \mid m = 2n+1\} \end{aligned}$$

   Alternatively, we can separately generate all strings of the form $\mathtt{0}^{\text{odd}}\mathtt{1}^*$, so that we don't have to worry about the case $\Delta = 1$ separately.

   $$\begin{aligned} S &\to D \mid M \mid L & \{\mathtt{0}^m\mathtt{1}^n \mid m \neq 2n\} \\ D &\to \mathtt{0} \mid \mathtt{00}D \mid D\mathtt{1} & \{\mathtt{0}^m\mathtt{1}^n \mid m \text{ is odd}\} \\ M &\to \mathtt{0}M \mid \mathtt{0}E & \{\mathtt{0}^m\mathtt{1}^n \mid m > 2n\} \\ L &\to L\mathtt{1} \mid E\mathtt{1} & \{\mathtt{0}^m\mathtt{1}^n \mid m < 2n \text{ and } m \text{ is even}\} \\ E &\to \varepsilon \mid \mathtt{00}E\mathtt{1} & \{\mathtt{0}^m\mathtt{1}^n \mid m = 2n\} \end{aligned}$$

   ■

Solution: Intuitively, we can parse any string $w \in L$ as follows. First, remove the first $2k$ $\mathtt{0}$s and the last k $\mathtt{1}$s, for the largest possible value of k. The remaining string cannot be empty, and it must consist entirely of $\mathtt{0}$s, entirely of $\mathtt{1}$s, or a single $\mathtt{0}$ followed by $\mathtt{1}$s.

$$\begin{aligned}
S &\to \mathtt{00}S\mathtt{1} \mid A \mid B \mid C & \{\mathtt{0}^m\mathtt{1}^n \mid m \neq 2n\}\\
A &\to \mathtt{0} \mid \mathtt{0}A & \mathtt{0}^+\\
B &\to \mathtt{1} \mid \mathtt{1}B & \mathtt{1}^+\\
C &\to \mathtt{0} \mid \mathtt{0}B & \mathtt{01}^*
\end{aligned}$$

■

3. $\{\mathtt{0},\mathtt{1}\}^* \setminus \{\mathtt{0}^{2n}\mathtt{1}^n \mid n \geq 0\}$

Solution: This language is the union of the previous language and the complement of $\mathtt{0}^*\mathtt{1}^*$, which is $(\mathtt{0}+\mathtt{1})^*\mathtt{10}(\mathtt{0}+\mathtt{1})^*$.

$$\begin{aligned}
S &\to T \mid X & \{\mathtt{0},\mathtt{1}\}^* \setminus \{\mathtt{0}^{2n}\mathtt{1}^n \mid n \geq 0\}\\
T &\to \mathtt{00}T\mathtt{1} \mid A \mid B \mid C & \{\mathtt{0}^m\mathtt{1}^n \mid m \neq 2n\}\\
A &\to \mathtt{0} \mid \mathtt{0}A & \mathtt{0}^+\\
B &\to \mathtt{1} \mid \mathtt{1}B & \mathtt{1}^+\\
C &\to \mathtt{0} \mid \mathtt{0}B & \mathtt{01}^*\\
X &\to Z\mathtt{10}Z & (\mathtt{0}+\mathtt{1})^*\mathtt{10}(\mathtt{0}+\mathtt{1})^*\\
Z &\to \varepsilon \mid \mathtt{0}Z \mid \mathtt{1}Z & (\mathtt{0}+\mathtt{1})^*
\end{aligned}$$

■

Work on these later:

4. $\{w \in \{\mathtt{0}, \mathtt{1}\}^* \mid \#(\mathtt{0}, w) = 2 \cdot \#(\mathtt{1}, w)\}$ — Binary strings where the number of 0s is exactly twice the number of 1s.

Solution: $S \to \varepsilon \mid SS \mid \mathtt{00}S\mathtt{1} \mid \mathtt{0}S\mathtt{1}S\mathtt{0} \mid \mathtt{1}S\mathtt{00}$.

Here is a sketch of a correctness proof; a more detailed proof appears in the homework.

For any string w, let $\Delta(w) = \#(\mathtt{0}, w) - 2 \cdot \#(\mathtt{1}, w)$. Suppose w is a binary string such that $\Delta(w) = 0$. Suppose w is nonempty and has no non-empty proper prefix x such that $\Delta(x) = 0$. There are three possibilities to consider:

- Suppose $\Delta(x) > 0$ for every proper prefix x of w. In this case, w must start with 00 and end with 1. Thus, $w = \mathtt{00}x\mathtt{1}$ for some string $x \in L$.
- Suppose $\Delta(x) < 0$ for every proper prefix x of w. In this case, w must start with 1 and end with 00. Let x be the shortest non-empty prefix with $\Delta(x) = 1$. Thus, $w = \mathtt{1}X\mathtt{00}$ for some string $x \in L$.
- Finally, suppose $\Delta(x) > 0$ for some prefix x and $\Delta(x') < 0$ for some longer proper prefix x'. Let x' be the shortest non-empty proper prefix of w with $\Delta < 0$. Then $x' = \mathtt{0}y\mathtt{1}$ for some substring y with $\Delta(y) = 0$, and thus $w = \mathtt{0}y\mathtt{1}z\mathtt{0}$ for some strings $y, z \in L$.

■

5. $\{\mathtt{0}, \mathtt{1}\}^* \setminus \{ww \mid w \in \{\mathtt{0}, \mathtt{1}\}^*\}$.

Solution: All strings of odd length are in L.

Let w be any even-length string in L, and let $m = |w|/2$. For some index $i \leq m$, we have $w_i \neq w_{m+i}$. Thus, w can be written as either $x\mathtt{1}y\mathtt{0}z$ or $x\mathtt{0}y\mathtt{1}z$ for some substrings x, y, z such that $|x| = i-1$, $|y| = m-1$, and $|z| = m-i$. We can further decompose y into a prefix of length $i-1$ and a suffix of length $m-i$. So we can write any even-length string $w \in L$ as either $x\mathtt{1}x'z'\mathtt{0}z$ or $x\mathtt{0}x'z'\mathtt{1}z$, for some strings x, x', z, z' with $|x| = |x'| = i-1$ and $|z| = |z'| = m-i$. Said more simply, we can divide w into two odd-length strings, one with a 0 at its center, and the other with a 1 at its center.

$S \to AB \mid BA \mid A \mid B$	strings not of the form ww
$A \to \mathtt{0} \mid \Sigma A \Sigma$	odd-length strings with 0 at center
$B \to \mathtt{1} \mid \Sigma B \Sigma$	odd-length strings with 1 at center
$\Sigma \to \mathtt{0} \mid \mathtt{1}$	single character

■

6. Prove that every regular language is context free.

Solution: It is in the notes. ■