

1. Let  $G = (V, E)$  be a graph. A set of edges  $M \subseteq E$  is said to be a matching if no two edges in  $M$  intersect at a vertex. A matching  $M$  is *perfect* if every vertex in  $V$  is incident to some edge in  $M$ ; alternatively  $M$  is perfect if  $|M| = |V|/2$  (which in particular implies  $|V|$  is even). See [Wikipedia article](#) for some example graphs and further background.

The PERFECTMATCHING problem is the following: does the given graph  $G$  have a perfect matching? This can be solved in polynomial time which is a fundamental result in combinatorial optimization with many applications in theory and practice. It turns out that the PERFECTMATCHING problem is easier to solve in *bipartite* graphs. A graph  $G = (V, E)$  is bipartite if its vertex set  $V$  can be partitioned into two sets  $L, R$  (left and right say) such that all edges are between  $L$  and  $R$  (in other words  $L$  and  $R$  are independent sets). Here is an attempted reduction from general graphs to bipartite graphs.

Given a graph  $G = (V, E)$  create a bipartite graph  $H = (V \times \{1, 2\}, E_H)$  as follows. Each vertex  $u$  is made into two copies  $(u, 1)$  and  $(u, 2)$  with  $V_1 = \{(u, 1) \mid u \in V\}$  as one side and  $V_2 = \{(u, 2) \mid u \in V\}$  as the other side. Let  $E_H = \{((u, 1), (v, 2)) \mid (u, v) \in E\}$ . In other words we add an edge between  $(u, 1)$  and  $(v, 2)$  iff  $(u, v)$  is an edge in  $E$ . Note that  $((u, 1), (u, 2))$  is not an edge in  $H$  for any  $u \in V$  since there are no self-loops in  $G$ .

Is the preceding reduction correct? To prove it is correct we need to check that  $H$  has a perfect matching if and only if  $G$  has one.

- Prove that if  $G$  has perfect matching then  $H$  has a perfect matching.
- Consider  $G$  to be  $K_3$  the complete graph on 3 vertices (a triangle). Show that  $G$  has no perfect matching but  $H$  has a perfect matching.
- Extend the previous example to obtain a graph  $G$  with an even number of vertices such that  $G$  has no perfect matching but  $H$  has.

Thus the reduction is incorrect although one of the directions is true.

**Solution:** • Suppose  $M$  is a perfect matching in  $G$ . We construct a perfect matching  $M'$  in  $H$  as follows. For each edge  $(u, v) \in M$  include the edges  $((u, 1), (v, 2))$  and  $((v, 1), (u, 2))$  in  $M'$ . It remains to verify that  $M'$  is a perfect matching. For each vertex  $u \in V$  there is exactly one edge  $(u, v) \in M$  incident to  $u$  since  $M$  is perfect. This means that in  $H$  there is exactly one edge  $((u, 1), (v, 2))$  incident to vertex  $(u, 1)$  in  $M'$  and also exactly one edge  $((v, 1), (u, 2))$  incident to  $(u, 2)$ . Thus  $M'$  is a perfect matching in  $H$ .

- Suppose the three vertices of  $G = K_3$  are labeled  $a, b, c$ . The edges of  $G$  are  $(a, b), (b, c), (c, a)$ . Then in  $H$  one can easily verify that the set of edges  $\{((a, 1), (b, 2)), ((b, 1), (c, 2)), ((c, 1), (a, 2))\}$  forms a perfect matching.  $G$  does not have a perfect matching since it has an odd number of vertices.
- Take  $G$  to be the disjoint union of two triangles, say  $G_1$  and  $G_2$ .  $G$  has 6 vertices which is even. From the preceding part one can see that the graph  $H$  obtained from the reduction has a perfect matching since we can take the perfect matching in  $H_1$  corresponding to  $G_1$  and the perfect matching in  $H_2$  corresponding to  $G_2$  and take their union.

■

2. An **independent set** in a graph  $G$  is a subset  $S$  of the vertices of  $G$ , such that no two vertices in  $S$  are connected by an edge in  $G$ . Suppose you are given a magic black box that somehow answers the following decision problem *in polynomial time*:
- INPUT: An undirected graph  $G$  and an integer  $k$ .
  - OUTPUT: TRUE if  $G$  has an independent set of size  $k$ , and FALSE otherwise.
- (a) Using this black box as a subroutine, describe algorithms that solves the following optimization problem *in polynomial time*:
- INPUT: An undirected graph  $G$ .
  - OUTPUT: The size of the largest independent set in  $G$ .

[Hint: You've seen this problem before.]

**Solution:** Suppose  $\text{INDSET}(V, E, k)$  returns TRUE if the graph  $(V, E)$  has an independent set of size  $k$ , and FALSE otherwise. Then the following algorithm returns the size of the largest independent set in  $G$ :

```

MAXINDSETSIZE( $V, E$ ):
  for  $k \leftarrow 1$  to  $V$ 
    if  $\text{INDSET}(V, E, k + 1) = \text{FALSE}$ 
      return  $k$ 

```

A graph with  $n$  vertices cannot have an independent set of size larger than  $n$ , so this algorithm must return a value. If  $G$  has an independent set of size  $k$ , then it also has an independent set of size  $k - 1$ , so the algorithm is correct.

The algorithm clearly runs in polynomial time. Specifically, if  $\text{INDSET}(V, E, k)$  runs in  $O((V + E)^c)$  time, then  $\text{MAXINDSETSIZE}(V, E)$  runs in  $O((V + E)^{c+1})$  time.

Yes, we could have used binary search instead of linear search. Whatever. ■

- (b) Using this black box as a subroutine, describe algorithms that solves the following search problem *in polynomial time*:
- INPUT: An undirected graph  $G$ .
  - OUTPUT: An independent set in  $G$  of maximum size.

**Solution (delete vertices):** I'll use the algorithm  $\text{MAXINDSETSIZE}(V, E)$  from part (a) as a black box instead. Let  $G - v$  denote the graph obtained from  $G$  by deleting vertex  $v$ , and let  $G - N(v)$  denote the graph obtained from  $G$  by deleting  $v$  and all neighbors of  $v$ .

```

MAXINDSET(G):
  S ← ∅
  k ← MAXINDSETSIZE(G)
  While G is not empty
    v is an arbitrary vertex of G
    if MAXINDSETSIZE(G - v) = k - |S|
      G ← G - v
    else
      G ← G - N(v)
      add v to S
  return S

```

Correctness of this algorithm follows inductively from the following claims:

**Claim 1.**  $\text{MAXINDSETSIZE}(G - v) = k$  if and only if  $G$  has an independent set of size  $k$  that excludes  $v$ .

**Proof:** Every independent set in  $G - v$  is also an independent set in  $G$ ; it follows that  $\text{MAXINDSETSIZE}(G - v) \leq k$ .

Suppose  $G$  has an independent set  $S$  of size  $k$  that does not include  $v$ . Then  $S$  is also an independent set of size  $k$  in  $G - v$ , so  $\text{MAXINDSETSIZE}(G - v)$  is at least  $k$ , and therefore equal to  $k$ .

On the other hand, suppose  $G - v$  has an independent set  $S$  of size  $k$ . Then  $S$  is also a maximum independent set of  $G$  (because  $|S| = k$ ) that excludes  $v$ .  $\square$

The algorithm clearly runs in polynomial time.  $\blacksquare$

**Solution (add edges):** I'll use the algorithm  $\text{MAXINDSETSIZE}(V, E)$  from part (a) as a black box instead. Let  $G + uv$  denote the graph obtained from  $G$  by adding edge  $uv$ .

```

MAXINDSET( $G$ ):
   $k \leftarrow \text{MAXINDSETSIZE}(G)$ 
  if  $k = 1$ 
    return any vertex
  for all vertices  $u$ 
    for all vertices  $v$ 
      if  $u \neq v$  and  $uv$  is not an edge
        if  $\text{MAXINDSETSIZE}(G + uv) = k$ 
           $G \leftarrow G + uv$ 
   $S \leftarrow \emptyset$ 
  for all vertices  $v$ 
    if  $\text{deg}(v) < V - 1$ 
      add  $v$  to  $S$ 
  return  $S$ 

```

The algorithm adds every edge it can without changing the maximum independent set size. Let  $G'$  denote the final graph. Any independent set in  $G'$  is also an independent set in the original input graph  $G$ . Moreover, the *largest* independent set in  $G'$  is also a largest independent set in  $G$ . Thus, to prove the algorithm correct, we need to prove the following claims about the final graph  $G'$ :

**Claim 2.** *The maximum independent set in  $G'$  is unique.*

**Proof:** Suppose the final graph  $G'$  has more than two maximum independent sets  $A$  and  $B$ . Pick any vertex  $u \in A \setminus B$  and any other vertex  $v \in A$ . The set  $B$  is still an independent set in the graph  $G' + uv$ . Thus, when the algorithm considered edge  $uv$ , it would have added  $uv$  to the graph, contradicting the assumption that  $A$  is an independent set.  $\square$

**Claim 3.** *Suppose  $k > 1$ . The unique maximum independent set of  $G'$  contains vertex  $v$  if and only if  $\text{deg}(v) < V - 1$ .*

**Proof:** Let  $S$  be the unique maximum independent set of  $G'$ , and let  $v$  be any vertex of  $G$ . If  $v \in S$ , then  $v$  has degree at most  $V - k < V - 1$ , because  $v$  is disconnected from every other vertex in  $S$ .

On the other hand, suppose  $\text{deg}(v) < V - 1$  but  $v \notin S$ . Then there must be at least vertex  $u$  such that  $uv$  is not an edge in  $G'$ . Because  $v \notin S$ , the set  $S$  is still an independent set in  $G' + uv$ . Thus, when the algorithm considered edge  $uv$ , it would have added  $uv$  to the graph, and we have a contradiction.  $\square$

The algorithm clearly runs in polynomial time.  $\blacksquare$

**To think about later:**

3. Formally, a **proper coloring** of a graph  $G = (V, E)$  is a function  $c: V \rightarrow \{1, 2, \dots, k\}$ , for some integer  $k$ , such that  $c(u) \neq c(v)$  for all  $uv \in E$ . Less formally, a valid coloring assigns each vertex of  $G$  a color, such that every edge in  $G$  has endpoints with different colors. The **chromatic number** of a graph is the minimum number of colors in a proper coloring of  $G$ .

Suppose you are given a magic black box that somehow answers the following decision problem *in polynomial time*:

- INPUT: An undirected graph  $G$  and an integer  $k$ .
- OUTPUT: TRUE if  $G$  has a proper coloring with  $k$  colors, and FALSE otherwise.

Using this black box as a subroutine, describe an algorithm that solves the following **coloring problem** *in polynomial time*:

- INPUT: An undirected graph  $G$ .
- OUTPUT: A valid coloring of  $G$  using the minimum possible number of colors.

[Hint: You can use the magic box more than once. The input to the magic box is a graph and **only** a graph, meaning **only** vertices and edges.]

**Solution:** First we build an algorithm to compute the minimum number of colors in any valid coloring.

```

CHROMATICNUMBER( $G$ ):
  for  $k \leftarrow V$  down to 1
    if COLORABLE( $G, k - 1$ ) = FALSE
      return  $k$ 

```

Given a graph  $G = (V, E)$  with  $n$  vertices  $v_1, v_2, \dots, v_n$ , the following algorithm computes an array  $color[1..n]$  describing a valid coloring of  $G$  with the minimum number of colors.

```

COLORING( $G$ ):
   $k \leftarrow$  CHROMATICNUMBER( $G$ )
  «— add a disjoint clique of size  $k$  —»
   $H \leftarrow G$ 
  for  $c \leftarrow 1$  to  $k$ 
    add vertex  $z_c$  to  $G$ 
    for  $i \leftarrow 1$  to  $c - 1$ 
      add edge  $z_i z_c$  to  $H$ 
  «— for each vertex, try each color —»
  for  $i \leftarrow 1$  to  $n$ 
    for  $c \leftarrow 1$  to  $k$ 
      add edge  $v_i z_c$  to  $H$ 
    for  $c \leftarrow 1$  to  $k$ 
      remove edge  $v_i z_c$  from  $H$ 
      if COLORABLE( $H, k$ ) = TRUE
         $color[i] \leftarrow c$ 
        break inner loop
      add edge  $v_i z_c$  from  $H$ 
  return  $color[1..n]$ 

```

In any  $k$ -coloring of  $H$ , the new vertices  $z_1, \dots, z_k$  must have  $k$  distinct colors, because every pair of those vertices is connected. We assign  $color[i] \leftarrow c$  to indicate that there is a  $k$ -coloring of  $H$  in which  $v_i$  has the same color as  $z_c$ . When the algorithm terminates,  $color[1..n]$  describes a valid  $k$ -coloring of  $G$ .

To prove that the algorithm is correct, we must prove that for all  $i$ , when the  $i$ th iteration of the outer loop ends,  $G$  has a valid  $k$ -coloring that is consistent with the partial coloring  $color[1..i]$ . Fix an integer  $i$ . The inductive hypothesis implies that when the  $i$ th iteration of the outer loop begins,  $G$  has a  $k$ -coloring consistent with the first  $i - 1$  assigned colors. (The base case  $i = 0$  is trivial.) If we connect  $v_i$  to every new vertex except  $z_c$ , then  $v_i$  must have the same color as  $z_c$  in any valid  $k$ -coloring. Thus, the call to COLORABLE inside the inner loop returns TRUE if and only if  $H$  has a  $k$ -coloring in which  $v_i$  has the same color as  $z_c$  (and the previous  $i - 1$  vertices are also colored). So COLORABLE must return TRUE during the second inner loop, which completes the inductive proof.

This algorithm makes  $O(kn) = O(n^2)$  calls to COLORABLE, and therefore runs in polynomial time. ■