

The following problems ask you to prove some “obvious” claims about recursively-defined string functions. In each case, we want a self-contained, step-by-step induction proof that builds on formal definitions and prior results, *not* on intuition. In particular, your proofs must refer to the formal recursive definitions of string length and string concatenation:

$$|w| := \begin{cases} 0 & \text{if } w = \varepsilon \\ 1 + |x| & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

$$w \cdot z := \begin{cases} z & \text{if } w = \varepsilon \\ a \cdot (x \cdot z) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

You may freely use the following results, which were proved in the lecture notes:

**Lemma 1:**  $w \cdot \varepsilon = w$  for all strings  $w$ .

**Lemma 2:**  $|w \cdot x| = |w| + |x|$  for all strings  $w$  and  $x$ .

**Lemma 3:**  $(w \cdot x) \cdot y = w \cdot (x \cdot y)$  for all strings  $w$ ,  $x$ , and  $y$ .

The *reversal*  $w^R$  of a string  $w$  is defined recursively as follows:

$$w^R := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ x^R \cdot a & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

For example, **STRESSED**<sup>R</sup> = **DESSERTS** and **WTF374**<sup>R</sup> = **473FTW**.

1. Prove that  $|w^R| = |w|$  for every string  $w$ .

**Solution (induction on  $w$ ):**

Let  $w$  be an arbitrary string.

Assume for any string  $x$  where  $|x| < |w|$  that  $|x^R| = |x|$ .

There are two cases to consider.

- If  $w = \varepsilon$ , then

$$\begin{aligned} |w^R| &= |\varepsilon| && \text{by definition of } ^R \\ &= |w| && \text{by definition of } |\cdot| \end{aligned}$$

- Otherwise,  $w = ax$  for some symbol  $a$  and some string  $x$ . In that case, we have

$$\begin{aligned} |w^R| &= |x^R \cdot a| && \text{by definition of } w^R \\ &= |x^R| + |a| && \text{by Lemma 2} \\ &= |x^R| + 1 && \text{by definition of } |\cdot| \text{ (twice)} \\ &= |x| + 1 && \text{by the induction hypothesis} = |w| \quad \text{by definition of } |\cdot| \end{aligned}$$

In both cases, we conclude that  $|w^R| = |w|$ . ■

2. Prove that  $(w \cdot z)^R = z^R \cdot w^R$  for all strings  $w$  and  $z$ .

**Solution (induction on  $w$ ):**

Let  $w$  and  $z$  be arbitrary strings.

Assume for any string  $x$  where  $|x| < |w|$  that  $(x \cdot z)^R = z^R \cdot x^R$ .

There are two cases to consider:

- If  $w = \varepsilon$ , then

$$\begin{aligned} (w \cdot z)^R &= z^R && \text{by definition of } \cdot \\ &= z^R \cdot \varepsilon && \text{by Lemma 1} \\ &= z^R \cdot w^R && \text{by definition of } ^R \end{aligned}$$

- Otherwise,  $w = ax$  for some symbol  $a$  and some string  $x$ .

$$\begin{aligned} (w \cdot z)^R &= (a \cdot (x \cdot z))^R && \text{by definition of } \cdot \\ &= (x \cdot z)^R \cdot a && \text{by definition of } ^R \\ &= (z^R \cdot x^R) \cdot a && \text{by the induction hypothesis, because } |x| < |w| \\ &= z^R \cdot (x^R \cdot a) && \text{by Lemma 3} \\ &= z^R \cdot w^R && \text{by definition of } ^R \end{aligned}$$

In both cases, we conclude that  $(w \cdot z)^R = z^R \cdot w^R$ . ■

But how did I know that the induction hypothesis needs to change the first string  $w$ , but not the second string  $z$ ? I wrote down the inductive argument first, and then noticed that in the proof for  $w \cdot z$ , we needed the inductive hypothesis on  $x \cdot z$ . Same string  $z$ , but  $w$  changed to  $x$ . Alternatively, in light of Lemma 2, I could have inducted on the **sum** of the string lengths with the inductive hypothesis "Assume for all strings  $x$  and  $y$  such that  $|x| + |y| < |w| + |z|$  that  $(x \cdot y)^R = x^R \cdot y^R$ ."

3. Prove that  $(w^R)^R = w$  for every string  $w$ .

**Solution (induction on  $w$ ):**

Let  $w$  be an arbitrary string.

Assume for any string  $x$  where  $|x| < |w|$  that  $(x^R)^R = x$ .

There are two cases to consider.

- If  $w = \varepsilon$ , then  $(w^R)^R = \varepsilon^R = \varepsilon$  by definition.
- Otherwise,  $w = ax$  for some symbol  $a$  and some string  $x$ .

$$\begin{aligned} (w^R)^R &= (x^R \cdot a)^R && \text{by definition of } ^R \\ &= a^R \cdot (x^R)^R && \text{by problem 2} \\ &= a \cdot (x^R)^R && \text{by definition of } ^R \\ &= a \cdot (x^R)^R && \text{by definition of } \cdot \\ &= a \cdot x && \text{by the induction hypothesis} \\ &= w && \text{by assumption} \end{aligned}$$

In both cases, we conclude that  $(w^R)^R = w$ . ■

**To think about later:** Let  $\#(a, w)$  denote the number of times symbol  $a$  appears in string  $w$ . For example,  $\#(\mathbf{X}, \mathbf{WTF374}) = 0$  and  $\#(\mathbf{0}, \mathbf{000010101010010100}) = 12$ .

4. Give a formal recursive definition of  $\#(a, w)$ .

**Solution:**

$$\#(a, w) = \begin{cases} 0 & \text{if } w = \varepsilon \\ 1 + \#(a, x) & \text{if } w = ax \text{ for some string } x \\ \#(a, x) & \text{if } w = bx \text{ for some symbol } b \neq a \text{ and some string } x \end{cases}$$

■

5. Prove that  $\#(a, w \cdot z) = \#(a, w) + \#(a, z)$  for all symbols  $a$  and all strings  $w$  and  $z$ .

**Solution (induction on  $w$ ):**

Let  $a$  be an arbitrary symbol, and let  $w$  and  $z$  be arbitrary strings.

Assume for any string  $x$  such that  $|x| < |w|$  that  $\#(a, x \cdot z) = \#(a, x) + \#(a, z)$ .

There are three cases to consider.

- If  $w = \varepsilon$ , then

$$\begin{aligned} \#(a, w \cdot x) &= \#(a, x) && \text{by definition of } \cdot \\ &= \#(a, w) + \#(a, x) && \text{by definition of } \# \end{aligned}$$

- If  $w = ax$  for some string  $x$ , then

$$\begin{aligned} \#(a, w \cdot z) &= \#(a, ax \cdot z) && \text{by definition of } \cdot \\ &= \#(a, a \cdot (x \cdot z)) && \text{by definition of } \cdot \\ &= 1 + \#(a, x \cdot z) && \text{by definition of } \# \\ &= 1 + \#(a, x) + \#(a, z) && \text{by the induction hypothesis} \\ &= \#(a, ax) + \#(a, z) && \text{by definition of } \# \\ &= \#(a, w) + \#(a, z) && \text{because } w = ax \end{aligned}$$

- If  $w = bx$  for some symbol  $b \neq a$  and some string  $x$ , then

$$\begin{aligned} \#(a, w \cdot z) &= \#(a, b \cdot (x \cdot z)) && \text{by definition of } \cdot \\ &= \#(a, x \cdot z) && \text{by definition of } \# \\ &= \#(a, x) + \#(a, z) && \text{by the induction hypothesis} \\ &= \#(a, bx) + \#(a, z) && \text{by definition of } \# \\ &= \#(a, w) + \#(a, z) && \text{because } w = bx \end{aligned}$$

In every case, we conclude that  $\#(a, w \cdot z) = \#(a, w) + \#(a, z)$ . ■

6. Prove that  $\#(a, w^R) = \#(a, w)$  for all symbols  $a$  and all strings  $w$ .

**Solution (induction on  $w$ ):** Let  $a$  be an arbitrary symbol, and let  $w$  be an arbitrary string.

Assume for any string  $x$  such that  $|x| < |w|$  that  $\#(a, x^R) = \#(a, x)$ .

There are three cases to consider.

- If  $w = \varepsilon$ , then  $w^R = \varepsilon = w$  by definition, so  $\#(a, w^R) = \#(a, w)$ .
- If  $w = ax$  for some string  $x$ , then

$$\begin{aligned}
 \#(a, w^R) &= \#(a, x^R \cdot a) && \text{by definition of } ^R \\
 &= \#(a, x^R) + \#(a, a) && \text{by problem 5} \\
 &= \#(a, x^R) + 1 && \text{by definition of } \# \\
 &= \#(a, x) + 1 && \text{by the induction hypothesis} \\
 &= \#(a, w) && \text{by definition of } \#
 \end{aligned}$$

- If  $w = bx$  for some symbol  $b \neq a$  and some string  $x$ , then

$$\begin{aligned}
 \#(a, w^R) &= \#(a, x^R \cdot b) && \text{by definition of } ^R \\
 &= \#(a, x^R) + \#(a, b) && \text{by problem 5} \\
 &= \#(a, x^R) && \text{by definition of } \# \\
 &= \#(a, x) && \text{by the induction hypothesis} \\
 &= \#(a, w) && \text{by definition of } \#
 \end{aligned}$$

In every case, we conclude that  $\#(a, w^R) = \#(a, w)$ . ■