Design Turing machines $M = (Q, \Sigma, \Gamma, \delta, \text{start}, \text{accept}, \text{reject})$ for each of the following tasks, either by listing the states Q, the tape alphabet Γ, and the transition function δ (in a table), or by drawing the corresponding labeled graph.

Each of these machines uses the input alphabet $\Sigma = \{\mathtt{1}, \mathtt{\#}\}$; the tape alphabet Γ can be any superset of $\{\mathtt{1}, \mathtt{\#}, \square, \triangleright\}$ where $\square$ is the blank symbol and $\triangleright$ is a special symbol marking the left end of the tape. Each machine should reject any input not in the form specified below.

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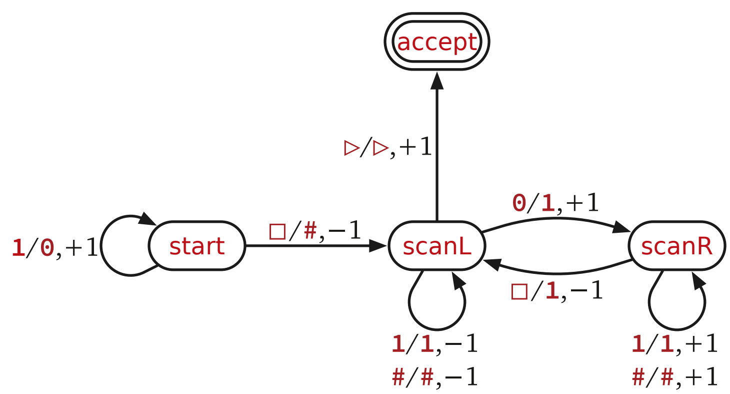
1. On input $\mathtt{1}^n$, for any non-negative integer n, write $\mathtt{1}^n\mathtt{\#}\mathtt{1}^n$ on the tape and accept.

Solution: Our Turing machine uses the tape alphabet $\Gamma = \{\mathtt{0}, \mathtt{1}, \mathtt{\#}, \square, \triangleright\}$ and the following states, in addition to accept and reject:

- start — Initialize the tape by replacing every **1** with **0**. When we find a blank, write **#** and start scanning left.
- scanL — Scan left for the rightmost **0**. If we find it, replace it with **1** and start scanning right. If we find $\triangleright$ instead, we're done; halt and accept.
- scanR — Scan right for the leftmost blank. When we find it, write **1** and start scanning left again.

Here is the transition graph of the machine. To simplify the drawing, we omit all transitions into the hidden reject state.



Here is the transition function; again, all unspecified transitions lead to the reject state.

$\delta($ p , $a)$	=	(q , b, Δ)	explanation
δ(start , **1**)	=	(start , **0**, +1)	init phase: replace **1**s with **0**s
δ(start , □)	=	(scanL , **#**, −1)	finished init phase; write **#** and start scanning left
δ(scanL, **1**)	=	(scanL , **1**, −1)	scan left to rightmost **0**
δ(scanL, **#**)	=	(scanL , **#**, −1)	
δ(scanL, **0**)	=	(scanR , **1**, +1)	found it; write **1** and start scanning right
δ(scanL, ▷)	=	(accept, ▷, +1)	found start of tape instead; we're done!
δ(scanR, **1**)	=	(scanR , **1**, +1)	main loop: scan right to leftmost □
δ(scanR, **#**)	=	(scanR , **#**, +1)	
δ(scanR, □)	=	(scanL , **1**, −1)	found it; write **1** and start scanning left

■

2. On input $\texttt{\#}^n\texttt{1}^m$, for any non-negative integers m and n, write $\texttt{1}^m$ on the tape and accept. In other words, delete all the #s and shift the 1s to the start of the tape.

3. On input $\texttt{\#1}^n$, for any non-negative integer n, write $\texttt{\#1}^{2n}$ on the tape and accept. *[Hint: Modify the Turing machine from problem 1.]*

4. On input $\texttt{1}^n$, for any non-negative integer n, write $\texttt{1}^{2^n}$ on the tape and accept. *[Hint: Use the three previous Turing machines as subroutines.]*

Questions to ponder:

- Think of a simple problem for which a 2-tape TM seems to offer much better efficiency than a 1-tape TM. Can you argue that 2-tape machine can be simulated by a 1-tape machine with only a quadratic slow down?
- Can you think about why having more than 2 tapes does not buy a lot of speed up? Can you argue why a k-tape TM can be simulated by a 2-tape TM with a slow down that has only only a poly-logarithmic overhead?
- How many bits does each *word* in your laptop/desktop have? How many bits did a desktop have 10 years ago, 20 years ago and 30 years ago? How does it limit the data you can work with?
- Suppose you want to multiply two n bit integers where $n = 10,000$. How would you write a program for it? What would be the time complexity?
- You may know about cryptography and RSA. The current RSA public key is 512 bits. Can you think of an algorithm to check if a given 512 bit number is a prime number? How many steps will it take?
- How can a RAM model with say 64 bits per word be simulated by a k-tape TM? What would be the slow down?