

Recall fooling sets and distinguishability. Two strings  $x, y \in \Sigma^*$  are suffix distinguishable with respect to a given language  $L$  if there is a string  $z$  such that exactly one of  $xz$  and  $yz$  is in  $L$ . This means that any DFA that accepts  $L$  must necessarily take  $x$  and  $y$  to different states from its start state. A set of strings  $F$  is a fooling set for  $L$  if *any* pair of strings  $x, y \in F, x \neq y$  are distinguishable. This means that any DFA for  $L$  requires at least  $|F|$  states. To prove non-regularity of a language  $L$  the standard way is to find an infinite fooling set  $F$  for  $L$ . Given a language  $L$  try to find a constant size fooling set first. Another way to prove that  $L$  is not regular is to show the following: for every  $n \geq 1$  prove that there is a fooling set of size at least  $n$ ; this may be helpful in thinking about the language.

Note that another method to prove non-regularity is via *reductions*. Suppose you want to prove that  $L$  is non-regular. You can do regularity preserving operations on  $L$  to obtain a language  $L'$  which you already know is non-regular. Then  $L$  must not have been regular. For instance if  $\bar{L}$  is not regular then  $L$  is also not regular. You will see an example in Problem 4 below.

Prove that each of the following languages is **not** regular.

1.  $\{0^{2n}1^n \mid n \geq 0\}$
2.  $\{0^m1^n \mid m \neq 2n\}$
3.  $\{0^{2^n} \mid n \geq 0\}$
4. Strings over  $\{0, 1\}$  where the number of 0s is exactly twice the number of 1s.
  - Describe an infinite fooling set for the language.
  - Use closure properties. What is language if you intersect the given language with  $0^*1^*$ ?
5. Strings of properly nested parentheses  $()$ , brackets  $[\ ]$ , and braces  $\{\}$ . For example, the string  $([\ ])\{\}$  is in this language, but the string  $([\ ])$  is not, because the left and right delimiters don't match.
  - Describe an infinite fooling set for the language.
  - Use closure properties.
6. Strings of the form  $w_1\#w_2\#\dots\#w_n$  for some  $n \geq 2$ , where each substring  $w_i$  is a string in  $\{0, 1\}^*$ , and some pair of substrings  $w_i$  and  $w_j$  are equal.

**Work on these later:**

7.  $\{0^{n^2} \mid n \geq 0\}$
8.  $\{w \in (0 + 1)^* \mid w \text{ is the binary representation of a perfect square}\}$