Given transition function \( \delta^* : Q \times \Sigma^* \rightarrow Q \), define its extended transition function inductively:

(i) \( \delta^*(q, \varepsilon) = q \)
(ii) \( \delta^*(q, x) = \delta^*(\delta(q, y), y) \) if \( x = ay \) with \( a \in \Sigma \) and \( y \in \Sigma^* \)

**Def:**

\( M \) accepts \( x \) iff \( \delta^*(q_0, x) \in A \).

Define \( L(M) = \{ x \in \Sigma^* : M \text{ accepts } x \} \).

**Exs:**

\( \Sigma = \{0, 1\} \).

a) all strings beginning with 001

b) all strings containing 001 as a substring

**Clarity:**
- Drop 6 lowest means 4 written HW probs + 2 GPSs
- \( q_3 \) found 001
- \( q_2 \) just seen 00 but not 001
- \( q_1 \) just seen 0 but not in \( q_2, q_3 \)
- \( q_0 \) none of above
c) all strings not containing 001

\[ q_0 \rightarrow q_1 \rightarrow q_2 \rightarrow q_3 \]

\( q_0 : \text{none of above} \)

d) all strings with even \# 0's and odd \# 1's

\[ \text{even}, \text{even} \rightarrow 0 \\
\text{even}, \text{odd} 1 \rightarrow 0 \\
\text{odd}, \text{even} 0 \rightarrow 1 \\
\text{odd}, \text{odd} 1 \rightarrow 0 \]

e) strings with length divisible by 5

\[ \text{0 mod 5} \rightarrow 0 \\
\text{1 mod 5} \rightarrow 1 \\
\text{2 mod 5} \rightarrow 0 \\
\text{3 mod 5} \rightarrow 1 \\
\text{4 mod 5} \rightarrow 0 \]

f) binary representation of all numbers divisible by 5

\( Q = \{0, 1, 2, 3, 4\} \)

eg. 1101, 1010, 1111, ...

\( +Z \text{ rewrite rule: } +2 \text{ GPS}_s \) (see course web page)
\[ Q = \{ 0, 1, 2, 3, 4 \} \\
A = \{ 0 \} \\
\delta(i, a) = (2i + a) \mod 5 \]

5 \times 2 + 1

9) all strings where 5th last symbol is 0.

\[ \text{e.g.} \quad 0110111000 \]

\[ Q = \{ \text{all strings of length 5} \} \quad |Q| = 32 \]

\[ S = 11111 \]

\[ \delta(a_1a_2a_3a_4a_5, a) = a_2a_3a_4a_5a \quad \forall a_1, a_5 \in \{ 0, 1 \} \]

\[ A = \{ a_1a_2a_3a_4a_5 : a_1 = 0 \} \]

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**Closure Properties**

**Thm:** If \( L \) is accepted by some DFA \( M \),

then its complement \( \overline{L} \) is also accepted

by some DFA \( M' \).

**Pf:** idea - take complement of accepting states

Given \( M = (Q, \Sigma, s, A, \delta) \),

\[ 101001 \]
Given \( M = (Q, \Sigma, s, A, \delta) \),

Construct \( M' = (Q, \Sigma, s', A', \delta) \)

where \( A' = Q \setminus A \)

Then \( x \in L(M') \iff \delta^*(s, x) \in A' \)

\( \iff \delta^*(s, x) \notin A \)

\( \iff x \notin L(M) = L \).

\[ \therefore L(M') = \overline{L}. \]

\[ \square \]

**Thm**

If \( L_1 \) is accepted by DFA \( M_1 \), \( L_2 \) " " " " " \( M_2 \),

then \( L_1 \cap L_2 \) is also accepted by some DFA \( M' \).

**PF:**

"remember" a pair of states

Given \( M_1 = (Q_1, \Sigma, s_1, A_1, \delta_1) \)

\( M_2 = (Q_2, \Sigma, s_2, A_2, \delta_2) \)

Construct \( M' = (Q', \Sigma, s', A', \delta') \).

where \( Q' = Q_1 \times Q_2 \)

\( s' = (s_1, s_2) \)

\( A' = A_1 \times A_2 = \{ (q_1, q_2) : q_1 \in A_1 \text{ and } q_2 \in A_2 \} \)

\[ \delta'( (q_1, q_2), \alpha ) = (\delta_1(q_1, \alpha), \delta_2(q_2, \alpha) ) \]

\[ \delta'( (q_1, q_2), x ) = (\delta_1^*(q_1, x), \delta_2^*(q_2, x) ) \]

**Lemma**

**PF:** by induction (omitted). \( \square \)
$$x \in L(M') \iff \delta^*((s_1, s_2), x) \in A_1 \times A_2$$

$$\iff (\delta^*(s, x), \delta^*_{A_2}(s_x))$$

$$\iff \delta^*(s_1, x) \in A_1 \text{ and } \delta^*_2(s_2, x) \in A_2$$

$$\iff x \in L(M_1) \text{ and } x \in L(M_2)$$

$$\iff x \in L_1 \cap L_2.$$ 

**Ex**

all strings containing 001 and having odd # 0's.

**M_1:**

**M_2:**

**M':**

\[\delta((q_1, \text{EVEN}), 0) = (\delta_1(q_1, 0), \delta_2(\text{EVEN}, 0)) = (q_2, \text{ODD})\]

**Cor**

If \( L_1 \) accepted by some DFA, then so is \( L_1 \cup L_2 \).

\( L_1 \cap L_2 \)

(\( L_1 \cap L_2 \))

(De Morgan's law)