Thursday, April 11, 2024 10:52 AM

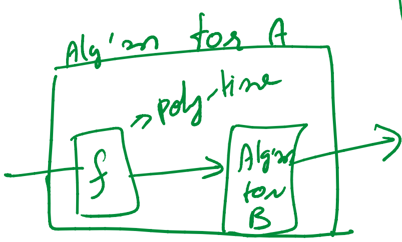
Q: How to show that a problem is hard to solve?
Has no efficient/poly-time alg'm?

$\boxed{\text{P: class/set of problems that have poly-time alg'm}}$

How to show our-problem $\notin$ P?

(eg. Finding closed walk $\rightarrow$ Finding SCC)

★ Reduction: $A \xrightarrow{\text{reduces to}} B$



2 Consequences.

① poly-time Alg'm for B gives poly-time Alg'm for A.

② No Alg'm for A $\Rightarrow$ No Alg'm for B.

So far, optimization Problems

O/P: Find max/min value / opt value.

↑ ↓ exe

Decision Problems:

Check if opt sol'n w/ value $\leq K$ exists?

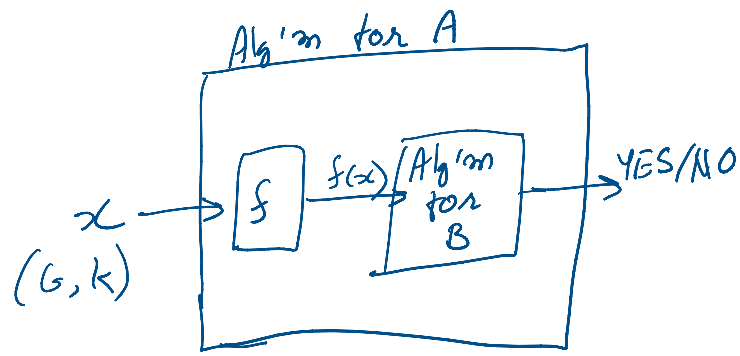
output : YES / NO

True / False

✩ Reduction:

$A \le_p B$ (A reduces to B in poly-time)

(A & B are decision problems)



f is a poly-time coputable function.

$\equiv$ f is a " alg'm. Runs in $O(poly(|x|))$

Composition of polynomials is a polynomial

($p(\cdot)$ & $q(\cdot)$ poly $\Rightarrow$ $p(q(\cdot))$ is a poly)

$\Downarrow$

Fact 1: poly time Alg'm for B $\Rightarrow$ poly-time Alg'm A.

$\neg$ ($B \in P \Rightarrow A \in P$)

$\Downarrow$

Fact 2: $A \notin P \Rightarrow B \notin P$

(no poly. time alg'm for A

$\Rightarrow$ NO poly-time alg'm for B)

Ex 1: Vertex Cover (VC) $\longrightarrow$ Set Cover (SC).

Input: A set U of elements.

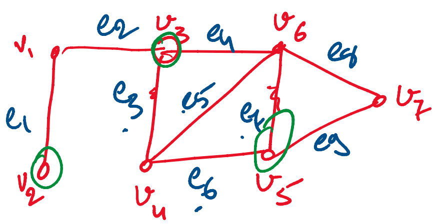
(VC) | (SC)

Input: Undirected graph $G=(V,E)$ & integer $k \geq 0$.

Output: YES iff $\exists$ VC of size $\leq k$ in G

iff $\exists S \subseteq V$, $|S| \leq k$ & $(u,v) \in E \Rightarrow u \in S$ or $v \in S$

e.g. G



$k=4$

$k=4$ $\{v_7, v_6, v_1, v_4\}$ o/p YES.

$k=3$ $\{ \quad \}$ NO

$\{v_2, v_3, v_5\}$ is an IS

Input: A set U of elements.
& subsets $A_1, \ldots, A_n \subseteq U$.
& integer $k' \geq 0$.

Output: YES iff $\exists$ a collection of size $\leq k'$ that covers the universe

iff $\exists I \subseteq \{1, \ldots, n\}$ s.t. $\bigcup_{i \in I} A_i = U$

e.g. $U = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$

$A_1 = \{e_1, e_2\}$, $A_2 = \{e_1\}$

$A_3 = \{e_3, e_4, e_2\}$, $A_4 = \{e_3, e_5, e_6\}$

$A_5 = \{e_6, e_7, e_9\}$, $A_6 = \{e_4, e_5, e_7, e_8\}$

$A_7 = \{e_8, e_9\}$ & $k' = k = 4$

Reduction: Vertex-Cover $\leq_p$ Set Cover

Given an instance of VC: $\boxed{G = (\overset{n}{V}, \overset{m}{E}) \ \& \ k \geq 0}$

f: Construct " " SC: U; $A_1, \ldots A_n \subseteq U$; k'

where $U = E$.

for each $v_i \in V$,
create $A_i = \{ e \mid e \text{ is incident on } v_i \text{ in } G \}$

& $k' = k$.

$O(mn)$ time

Correctness Pf:

... of size $\leq k \iff \exists$ SC, $I \subseteq \{1, \ldots, n\}$ ...

Correctness Pf:

Claim: G has a VC S of size $\le k \iff \exists$ SC, $I \subseteq \{1, \dots, n\}$ of size $\le k'$

Pf: $(\Rightarrow)$ $\boxed{I = \{i \mid v_i \in S'\}}$

$$\bigcup_{i \in I} A_i = \bigcup_{i \in I} \{e \mid e \text{ incident on } v_i \text{ in } G\}$$

$$= \bigcup_{v_i \in S'} \{e \mid e \text{ " " }\}$$

$$= E = U$$

$(\Leftarrow)$ Given $I \subseteq \{1, \dots, n\} \rightarrow S' = \{v_i \mid i \in I\}$

sol'n of S.C.

S' covers all edges.

$$\bigcup_{v_i \in S'} \{e \mid e \text{ incident on } v_i\}$$

$$= \bigcup_{i \in I} \{e \mid e \text{ " " "}\}$$

$$= \bigcup_{i \in I} A_i = U = E$$

Ex 2: Independent set (IS) $\le_p$ Vertex cover.

IS:

Input: Given undirect graph $G = (V, E)$, int $k \ge 0$

O/P: YES iff $\exists$ IS of size $\ge k$ in G.

iff $\exists S \subseteq V$, $|S| \ge k$

iff $\exists S \subseteq V$, $|S| \ge k$

& $\neg(\forall u, v \in S \Rightarrow (u,v) \notin E)$

iff $\forall (u,v) \in E \Rightarrow u \notin S$ or $v \notin S$

$\Rightarrow u \in V \setminus S$ or $v \in V \setminus S$

$\Rightarrow V \setminus S$ is a VC.

Reduction: $IS \le_p VC$.

Given i/p of IS : $G = (V, E)$ & $k \ge 0$

Construct " " VC : $G' = (V', E')$ & $k' \ge 0$

where $G' = G$.

$k' = (n - k)$

Claim: G has IS S of size $\ge k \iff G'$ has VC S' of size $\le k' = (n-k)$

Pf: $\boxed{S' = V \setminus S}$

$IS \le_p VC \le_p SC$

$\Downarrow$

$IS \le_p SC$? YES!

How to find the first hard problem?