Monday, April 1, 2024 2:22 PM

## Announcements:

① Midterm 2 on April 9, Tuesday

② Conflict " " " 8, Monday

(Form fill up deadline today at 2 pm)

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## Greedy Alg'm:

For solving optimization problems.

Incrementally build soln.

Making "local" optimal choice / greedy choice in every step / iteration.

Adv: Simple & Fast

Disadv: They can be wrong!

Proof of correctness is needed.

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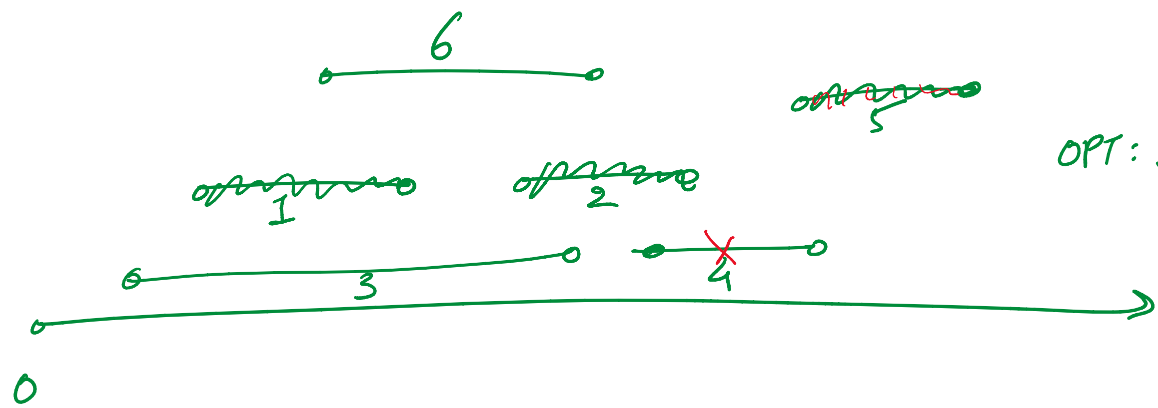
Ex 1: Interval / Job scheduling.

Given $n$ intervals (jobs) $[s_1, f_1]$, $[s_2, f_2]$ ... , $[s_n, f_n]$

start time finish time.

Find maximum # non-overlapping intervals.

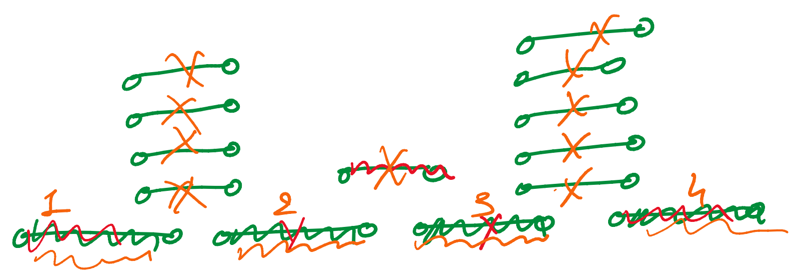
eg




OPT: 3

idea 1: Pick job of earliest start time
{3, 4} fails!

idea 2: Pick job that has min # intersections
{5, 2, 1} yay!




idea 3: Pick smallest job fails!

idea 4: Earliest end time.

WORKS!

## Greedy Alg'm:

1. repeat {
2. Pick $[s_i, t_i]$ w/ smallest $t_i$ among the intervals left. Add this to the sol'n
3. Remove all intervals intersecting w/ $[s_i, t_i]$
4. } Until no intervals left.

$O(n)$

Running Time: $O(n^2)$

Better Run time: Sort in ascending order of end times $O(n \log n)$

$+ \; O(n)$ work.

$= O(n \log n)$.

## Proof of Correctness:

O/P of Alg be $I$.

Let $I^*$ be an opt. sol$^n$ (unknown)

Let the first interval in $I^*$ be $[s^*, f^*]$

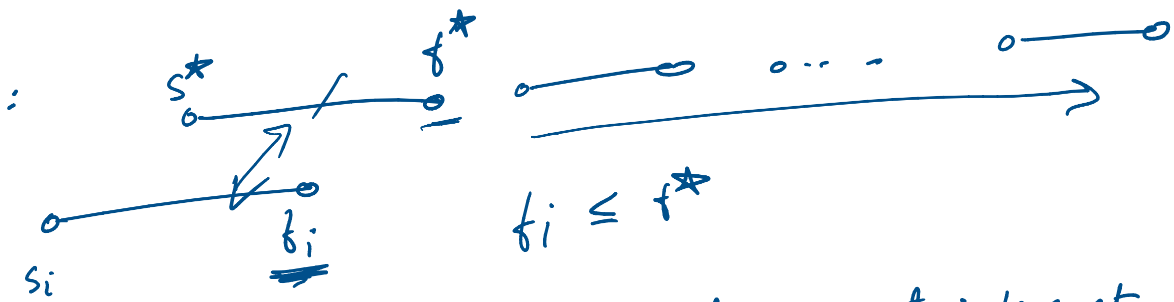
Let the first interval in $I$ be $[s_i, \underline{t_i}]$.

Let the first

(by choice of greedy Alg'm)

We know that

$I^*$ :




$f_i \le f^*$

$\Rightarrow [s_i, f_i]$ does not intersect w/ any interval in $I^* \setminus \{[s^*, f^*]\}$.

$$\boxed{\tilde{I}^* \leftarrow I^* \setminus \{[s^*, f^*]\} \cup \{[s_i, f_i]\}}$$ is a feasible sol'n

$|\tilde{I}^*| = |I^*|$

(exchange argument) & it is also optimum.

Remove $[s_i, f_i]$ & all it's intersecting intervals
$\downarrow$
& Repeat $\rightarrow$ Induction. $\leftarrow$ smaller instance.

Rmk: Doesn't work w/ weights. (DP).

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Ex 2: Job scheduling to minimize the total wait time.

Given $n$ jobs w/ processing times

$$\underline{P_1, P_2, P_3, \ldots}, P_n \ge 0$$

want to find an ordering of the jobs

want to find an ordering of the jobs that minimizes the total wait time.

$$\text{cost} = 0 + P_1 + (P_1 + P_2) + \dots + (P_1 + P_2 + \dots + P_{n-1})$$

Ordering that Minimizes the cost.

e.g.

| jobs | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Processing time ($P_i$) | 3 | 4 | 1 | 8 | 2 |

$$\text{cost}(3,4,1,8,2) = 0 + 3 + (3+4) + (3+4+1) + (3+4+1+8)$$
$$= 34$$

$$\text{cost}(3,4,1,2,8) = 0 + 3 + (3+4) + (3+4+1) + (3+4+1+2)$$
$$= 28$$

$$\text{cost}(3,1,4,2,8) = 0 + 3 + \dots$$
$$= 25$$

Claim: Order in increasing order

**Greedy Alg'm:** Order in increasing order of processing time.

$$\text{cost}(1, 2, 3, 4, 8) = 0 + 1 + (1+2) + (1+2+3) + (1+2+3+4)$$

$$= 20$$

**Correctness Proof:**

Let $p_1^*, p_2^*, \ldots p_i^*, p_{i+1}^*, p_n^*$ is the optimal ordering & not in sorted order

$\Rightarrow \exists i.$ $\boxed{p_i^* > p_{i+1}^*}$

$$\text{opt-cost} = \cancel{0} + \cancel{p_1^*} + \cancel{(p_1^* + p_2^*)} + \cdots + \cancel{(p_1^* + p_2^* + \cdots p_{i-1}^* + p_i^*)}$$
$$+ (\cancel{p_1^* + p_2^* + \cdots + p_i^*} + p_{i+1}^*) + \cdots +$$

Consider new ordering: $p_1^*, p_2^*, \ldots p_{i-1}^*, p_{i+1}^*, p_i^*, p_{i+2}^*, \ldots, p_n^*$

$$\text{New cost} = \cancel{0} + \cancel{p_1^*} + \cancel{(p_1^* + p_2^*)} + \cdots + (\cancel{p_1^* + p_2^* + \cdots + p_{i-1}^*} + p_{i+1}^*)$$
$$+ (\cancel{p_1^* + p_2^* + \cdots + p_{i-1}^* + p_{i+1}^*} + p_i^*) + \cdots$$

$$\text{opt-cost} - \text{New cost} = p_i^* - p_{i+1}^* > 0 \quad (\because p_i^* > p_{i+1}^*)$$

$\Rightarrow$ opt-cost $>$ new-cost

contradiction to optimality of $p_1^*, \ldots, p_n^*$
|||
min cost. ∎

contradiction ... min-cost. ∎

## Extends to weighted problem:

We are also given $w_i \geq 0$ $\forall i$ together with $p_i$ (processing time).

$$\overset{w_1}{p_1}, \overset{w_2}{p_2}, \dots \overset{w_n}{p_n}$$

$$cost = w_1 \cdot 0 + w_2(p_1) + w_3(p_1+p_2) + w_4(p_1+p_2+p_3) + \dots + w_n(p_1+p_2+\dots+p_{n-1})$$

Find ordering that minimizes the cost.

Intuition: increasing order of $p_i$
decreasing order of $w_i$ (weight / priority)

↓

increasing order of $\frac{p_i}{w_i}$

Greedy Alg'm: Increasing order of $\frac{p_i}{w_i}$

Correctness pf: (Similar to unweighted)

OPT: $p_1^*, p_2^*, \dots, p_i^*, p_{i+1}^*, \dots, p_n^*$

Correctness :

$$OPT: \; p_1^*, p_2^*, \dots, p_i^*, p_{i+1}^*, \dots, p_n^*$$

$$\exists i: \quad \frac{p_i^*}{w_i^*} > \frac{p_{i+1}^*}{w_{i+1}^*}$$

$$\text{after swap}: \; \underline{p_1^*, p_2^*, \dots, p_{i-1}^*}, \; \underline{p_{i+1}^*}, p_i^*, p_{i+2}^*, \dots, p_n^*$$

$w_{i+1}^*$

$$\text{OPT-cost}: \; \cancel{w_1^*(0) + w_2^*(p_1^*) + \dots} + w_i^*(\cancel{p_1^* + \dots + p_{i-1}^*}) + w_{i+1}^*(\cancel{p_1^* + p_2^* + \dots + p_{i-1}^*} + p_i^*) + \cancel{w_{i+2}^*(p_1^* + \dots + p_{i+1}^*) + \dots}$$

$-$

$$\text{New-cost (after swap)}: \; \cancel{w_1^*(0) + w_2^*(p_1^*) + \dots} + w_{i+1}^*(\cancel{p_1^* + \dots + p_{i-1}^*}) + w_i^*(\cancel{p_1^* + p_2^* + \dots + p_{i-1}^*} + p_{i+1}^*) + \cancel{w_{i+2}^*(p_1^* + \dots + p_{i-1}^* + p_{i+1}^* + p_i^*) + \dots}$$

$$\text{Opt-cost} - \text{New-cost} = w_{i+1}^* \cdot p_i^* - w_i^* \, p_{i+1}^* > 0$$

$$\because \; \frac{p_i^*}{w_i^*} > \frac{p_{i+1}^*}{w_{i+1}^*}$$

∎