

Proving that a problem  $X$  is NP-hard requires several steps:

- Choose a problem  $Y$  that you already know is NP-hard (because we told you so in class).
- Describe an algorithm to solve  $Y$ , using an algorithm for  $X$  as a subroutine. Typically this algorithm has the following form: Given an instance of  $Y$ , transform it into an instance of  $X$ , and then call the magic black-box algorithm for  $X$ .
- **Prove** that your algorithm is correct. This always requires two separate steps, which are usually of the following form:
  - **Prove** that your algorithm transforms “good” instances of  $Y$  into “good” instances of  $X$ .
  - **Prove** that your algorithm transforms “bad” instances of  $Y$  into “bad” instances of  $X$ . Equivalently: Prove that if your transformation produces a “good” instance of  $X$ , then it was given a “good” instance of  $Y$ .
- Argue that your algorithm for  $Y$  runs in polynomial time.

**1** A **Hamiltonian cycle** in a graph  $G$  is a cycle that goes through every vertex of  $G$  exactly once. Deciding whether an arbitrary graph contains a Hamiltonian cycle is NP-hard.

A **tonian cycle** in a graph  $G$  is a cycle that goes through at least *half* of the vertices of  $G$ . Prove that deciding whether a graph contains a tonian cycle is NP-hard.

**2** *Big Clique* is the following decision problem: given a graph  $G = (V, E)$ , does  $G$  have a clique of size at least  $n/2$  where  $n = |V|$  is the number of nodes? Prove that *Big Clique* is NP-hard.

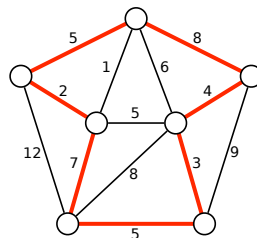
**3** Recall the following  $k$ COLOR problem: Given an undirected graph  $G$ , can its vertices be colored with  $k$  colors, so that every edge touches vertices with two different colors?

**3.A.** Describe a direct polynomial-time reduction from 3COLOR to 4COLOR.

**3.B.** Prove that  $k$ COLOR problem is NP-hard for any  $k \geq 3$ .

**To think about later:**

**4** Let  $G$  be an undirected graph with weighted edges. A Hamiltonian cycle in  $G$  is **heavy** if the total weight of edges in the cycle is at least half of the total weight of all edges in  $G$ . Prove that deciding whether a graph contains a heavy Hamiltonian cycle is NP-hard.



A heavy Hamiltonian cycle. The cycle has total weight 34; the graph has total weight 67.