Prove that each of the following languages is not regular.

1. \{0^{2^n} \mid n \geq 0\}

Solution:

Choose \( F = \{0^{2^n} \mid n \geq 0\} \).

Let \( x \) and \( y \) be two arbitrary strings of \( F \) with \( x \neq y \).

Then \( x = 0^{2^i} \) and \( y = 0^{2^j} \) for some non-negative integers \( i \neq j \).

Choose \( z = 0^{2^i} \).

Then \( xz = 0^{2^i}0^{2^i} = 0^{2^{i+1}} \in L \).

And \( yz = 0^{2^j}0^{2^i} = 0^{2^i+2^j} \notin L \), because \( i \neq j \) (since \( 2^i + 2^j \) cannot be a power of 2).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

2. \{0^{2n}1^n \mid n \geq 0\}

Solution:

Choose \( F = \{0^i \mid i \geq 0\} \).

Let \( x \) and \( y \) be two arbitrary strings in \( F \) with \( x \neq y \).

Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).

Choose \( z = 0^i1^i \).

Then \( xz = 0^{2i}1^i \in L \).

And \( yz = 0^{i+j}1^i \notin L \), because \( i + j \neq 2i \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

3. \{0^{m1^n} \mid m \neq 2n\}

Solution:

Choose \( F = \{0^i \mid i \geq 0\} \).

Let \( x \) and \( y \) be two arbitrary strings in \( F \) with \( x \neq y \).

Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).

Choose \( z = 0^i1^i \).

Then \( xz = 0^{2i}1^i \notin L \).

And \( yz = 0^{i+j}1^i \in L \), because \( i + j \neq 2i \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.
Strings over \{0, 1\} where the number of 0s is exactly twice the number of 1s.

**Solution:**
Choose \( F = \{0^i \mid i \geq 0\} \).
Let \( x \) and \( y \) be two arbitrary strings in \( F \) with \( x \neq y \).
Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).
Choose \( z = 0^i 1^i \).
Then \( xz = 0^{2i} 1^i \in L \).
And \( yz = 0^{i+j} 1^i \notin L \), because \( i + j \neq 2i \).
Thus, \( F \) is a fooling set for \( L \).
Because \( F \) is infinite, \( L \) cannot be regular.

**Solution:**
If \( L \) were regular, then the language
\[
((0 + 1)^* \setminus L) \cap 0^*1^* = \{0^m 1^n \mid m \neq 2n\}
\]
would also be regular, because regular languages are closed under complement and intersection. But we just proved that \( \{0^m 1^n \mid m \neq 2n\} \) is not regular in problem 3. [This proof would be worth full credit in homework or an exam, if we do not explicitly specify that you should use the fooling set method.]

Strings of properly nested parentheses (), brackets [], and braces {}. For example, the string \(([]\})\) is in this language, but the string \(( []\})\) is not, because the left and right delimiters don’t match.

**Solution:**
Choose \( F = \{ (^i \mid i \geq 0\} \).
Let \( x \) and \( y \) be two arbitrary strings in \( F \) with \( x \neq y \).
Then \( x = (^i \) and \( y = (^j \) for some non-negative integers \( i \neq j \).
Choose \( z = ^i \).
Then \( xz = (^i)^i \in L \).
And \( yz = (^j)^i \notin L \), because \( i \neq j \).
Thus, \( F \) is a fooling set for \( L \).
Because \( F \) is infinite, \( L \) cannot be regular.

Strings of the form \( w_1 \# w_2 \# \cdots \# w_n \) for some \( n \geq 2 \), where each substring \( w_i \) is a string in \( \{0, 1\}^* \), and some pair of substrings \( w_i \) and \( w_j \) are equal.
Solution:

Choose \( F = \{0^i \mid i \geq 0\} \).

Let \( x \) and \( y \) be arbitrary strings in \( F \) with \( x \neq y \).

Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).

Choose \( z = \#0^i \).

Then \( xz = 0^i \#0^i \in L \).

And \( yz = 0^j \#0^i \notin L \), because \( i \neq j \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

Extra problems

7 \( \{ w \in (0 + 1)^* \mid w \text{ is the binary representation of a perfect square} \} \)

Solution:

Idea: We design our fooling set around numbers of the form \((2^k + 1)^2 = 2^{2k} + 2^{k+1} + 1\), which has binary representation \(10^{k-2}10^k1\). The argument is somewhat simpler if we further restrict \( k \) to be even.

Choose \( F = \{10^{2i}1 \mid i \geq 0\} \).

Let \( x \) and \( y \) be two distinct arbitrary strings in \( F \).

Then \( x = 10^{2i-2}1 \) and \( y = 10^{2j-2}1 \), for some positive integers \( i \neq j \). Without loss of generality, assume \( i < j \). (Otherwise, swap \( x \) and \( y \).)

Choose \( z = 0^{2i}1 \).

Then \( xz = 10^{2i-2}10^{2i}1 \) is the binary representation of \( 2^{4i} + 2^{2i+1} + 1 = (2^{2i} + 1)^2 \), and therefore \( xz \in L \).

On the other hand, \( yz = 10^{2j-2}10^{2i}1 \) is the binary representation of \( 2^{2i+2j} + 2^{2i+1} + 1 \). Simple algebra gives us the inequalities

\[
(2^{i+j})^2 = 2^{2i+2j} < 2^{2i+2j} + 2^{2i+1} + 1 < 2^{2(i+j)} + 2^{i+j+1} + 1 = (2^{i+j} + 1)^2.
\]

So \( 2^{2i+2j} + 2^{2i+1} + 1 \) lies between two consecutive perfect squares, and thus is not a perfect square, which implies that \( yz \notin L \).

We conclude that \( F \) is a fooling set for \( L \). Because \( F \) is infinite, \( L \) cannot be regular.