1. Let $L$ be an arbitrary regular language. Prove that the language $\text{reverse}(L) := \{ w^R \mid w \in L \}$ is regular. 

*Hint:* Consider a DFA $M$ that accepts $L$ and construct an NFA that accepts $\text{reverse}(L)$.

2. Let $L$ be an arbitrary regular language. Prove that the language $\text{insert1}(L) := \{ xy \mid xy \in L \}$ is regular.

Intuitively, $\text{insert1}(L)$ is the set of all strings that can be obtained from strings in $L$ by inserting exactly one $1$. For example, if $L = \{ \varepsilon, OOK! \}$, then $\text{insert1}(L) = \{ 1, 1OOK!, 01OK!, OO1K!, OOK1!, OOK!1 \}$.

*Work on these later:*

3. Let $L$ be an arbitrary regular language. Prove that the language $\text{delete1}(L) := \{ xy \mid x1y \in L \}$ is regular.

Intuitively, $\text{delete1}(L)$ is the set of all strings that can be obtained from strings in $L$ by deleting exactly one $1$. For example, if $L = \{ 101101, 00, \varepsilon \}$, then $\text{delete1}(L) = \{ 01101, 10101, 10110 \}$.

4. Consider the following recursively defined function on strings:

$$
\text{stutter}(w) := \begin{cases} 
\varepsilon & \text{if } w = \varepsilon \\
aa \cdot \text{stutter}(x) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x
\end{cases}
$$

Intuitively, $\text{stutter}(w)$ doubles every symbol in $w$. For example:

- $\text{stutter}(\text{PRESTO}) = \text{PPRREESSTTOO}$
- $\text{stutter}(\text{HOCUS\&POCUS}) = \text{HHOOCUSS\&PPOOCUSS}$

Let $L$ be an arbitrary regular language.

1. Prove that the language $\text{stutter}^{-1}(L) := \{ w \mid \text{stutter}(w) \in L \}$ is regular.
2. Prove that the language $\text{stutter}(L) := \{ \text{stutter}(w) \mid w \in L \}$ is regular.

5. Consider the following recursively defined function on strings:

$$
\text{evens}(w) := \begin{cases} 
\varepsilon & \text{if } w = \varepsilon \\
\varepsilon & \text{if } w = a \text{ for some symbol } a \\
b \cdot \text{evens}(x) & \text{if } w = abx \text{ for some symbols } a \text{ and } b \text{ and some string } x
\end{cases}
$$

Intuitively, $\text{evens}(w)$ skips over every other symbol in $w$. For example:

- $\text{evens}(\text{EXPPELLIARMUS}) = \text{XELAMS}$
- $\text{evens}(\text{AVADA\&KEDAVRA}) = \text{VD\&EAR}$

Once again, let $L$ be an arbitrary regular language.

1. Prove that the language $\text{evens}^{-1}(L) := \{ w \mid \text{evens}(w) \in L \}$ is regular.
2. Prove that the language $\text{evens}(L) := \{ \text{evens}(w) \mid w \in L \}$ is regular.