Problem Old.1.1: Let \( L \subseteq \{0, 1\}^* \) be a language defined recursively as follows:

- \( \varepsilon \in L \).
- For all \( w \in L \) we have \( 0w1 \in L \).
- For all \( x, y \in L \) we have \( xy \in L \).
- And these are all the strings that are in \( L \).

Prove, by induction, that for any \( w \in L \), and any prefix \( u \) of \( w \), we have that \( \#_0(u) \geq \#_1(u) \). Here \( \#_0(u) \) is the number of 0 appearing in \( u \) (\( \#_1(u) \) is defined similarly). You can use without proof that \( \#_0(xy) = \#_0(x) + \#_0(y) \), for any strings \( x, y \).

Solution:

Proof. The proof is by induction on the length of \( w \).

Base case: If \( |w| = 0 \) then \( w = \varepsilon \), and then \( \#_0(w) = 0 \geq \#_1(u) = 0 \). Since the only prefix of the empty string is itself, the claim readily follows.

Induction hypothesis: Assume that the claim holds for all strings of length \( < n \).

Induction step: We need to prove the claim for a string \( w \) of length \( n \). There are two possibilities:

- \( w = 0z1 \), for some string \( z \in L \).
  Let \( u \) be any prefix of \( w \). If \( u = \varepsilon \) or \( u = 0 \) then the claim clearly holds for \( u \).
  If \( u = w \), then
  \[ \#_0(u) = \#_0(w) = 1 + \#_0(z) + 0 \geq 1 + \#_1(z) = \#_1(w) = \#_1(u), \]
  which implies the claim (we used the induction hypothesis on \( z \), since \( z \in L \) and \( |z| = |w| - 2 < n \)).
  So the remaining case is when \( u = 0z' \), where \( z' \) is a prefix of \( z \). In this case,
  \[ \#_0(u) = \#_0(0z') = 1 + \#_0(z') \geq 1 + \#_1(z') = 1 + \#_1(u) > \#_1(u), \]
  Again, we used the induction hypothesis on \( z \), since \( z \in L \), \( z' \) is a prefix of \( z \), and \( z \) strictly shorter than \( w \). This implies the claim.

- \( w = xy \), for some strings \( x, y \in L \), such that \( |x|, |y| > 0 \).
  Let \( u \) be a prefix of \( w \). If \( u \) is a prefix of \( x \), then the claim holds readily by induction.
  The remaining case is when \( u = xz \), for some \( z \) which is prefix of \( y \). Here,
  \[ \#_0(u) = \#_0(xz) = \#_0(x) + \#_0(z) \geq \#_1(x) + \#_1(z) = \#_1(u), \]
  by using the induction hypothesis on \( x \) (which is a prefix of itself), and on \( z \) (which is a prefix of \( y \)), noting that both \( x \) and \( y \) are strictly shorter than \( w \).

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Problem Old.1.2: Consider the recurrence

\[ T(n) = \begin{cases} 
  T(\lfloor n/3 \rfloor) + T(\lfloor n/4 \rfloor) + T(\lfloor n/5 \rfloor) + T(\lfloor n/6 \rfloor) + n & n \geq 6 \\
  1 & n < 6 
\end{cases} \]

Prove by induction that \( T(n) = O(n) \).

Solution:

Claim 1. For \( c \geq 20 \), and for all \( n \geq 1 \), we have \( T(n) \leq cn \).

Proof. Base case. For \( n < 6 \) the claim holds for any \( c \geq 1 \) by definition.

Induction hypothesis. Let \( n \geq 6 \). Assume that \( T(k) \leq ck \) for all \( 1 \leq k < n \).

Induction step. We need to prove that \( T(n) \leq cn \). We know that

\[
T(n) = T(\lfloor n/3 \rfloor) + T(\lfloor n/4 \rfloor) + T(\lfloor n/5 \rfloor) + T(\lfloor n/6 \rfloor) + n
\leq c \lfloor n/3 \rfloor + c \lfloor n/4 \rfloor + c \lfloor n/5 \rfloor + c \lfloor n/6 \rfloor + n \quad \text{(by the induction hypothesis)}
\leq cn/3 + cn/4 + cn/5 + cn/6 + n
\leq \left( \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \right) cn + n = \left( \frac{3}{4} + \frac{1}{5} \right) cn + n = \left( \frac{19}{20}c + 1 \right) n \leq cn,
\]

provided that

\[ \frac{19}{20}c + 1 \leq c \iff 1 \leq \frac{1}{20}c \iff c \geq 20. \]

IMPORTANT NOTE: make sure that the “\( c \)” in the conclusion from the induction step \( (T(n) \leq cn) \) is the same as the “\( c \)” you start with from the induction hypothesis \( (T(k) \leq ck \) for \( k < n \)). If not (for example, if you could only conclude that \( T(n) \leq 1.01cn \)), then the whole proof would be incorrect—because the constant factor will “blow up” when we repeat! (General advice: avoid big-O notation inside induction proofs!)