CS/ECE 374 A (Spring 2024)
Homework 10 (due Apr 18 Thursday at 10am)

Instructions: As in previous homeworks.

Problem 10.1: Consider the following problem: given a set S of n distinct positive integers and a number L, we want to form the largest number of disjoint pairs in S, such that the sum of each pair is at most L.

For example, for $S = \{1, 3, 4, 5, 8\}$ and $L = 10$, an optimal solution has 2 pairs (e.g., one optimal solution is $\{1, 8\}$ and $\{4, 5\}$, but there are a number of other solutions that are also optimal).

Consider the following greedy strategy:

repeat {
    find a pair $\{a, b\} \subseteq S$ with $a + b \leq L$ maximizing $a + b$
    if no such pair exists then stop
    else output the pair $\{a, b\}$, and remove a and b from S
}

(a) (85 pts) Prove that this algorithm always correctly finds an optimal solution.

Like various greedy correctness proofs from class, you should use an "exchange" argument: let $\{a, b\} \subseteq S$ with $a + b \in L$ maximizing $a + b$; suppose P^* is an optimal solution, and suppose $\{a, b\}$ is not in P^*; show how to modify P^* to get a different feasible solution that is just as good and that uses $\{a, b\}$...

(b) (15 pts) Consider a variant of the problem, where we want to find the largest number of disjoint triples, such that the sum of each triple is at most L. Show that the same greedy strategy (where now we find a triple $\{a, b, c\} \subseteq S$ with $a + b + c \leq L$ maximizing $a + b + c$ at each iteration) does not always correctly find an optimal solution. You should give a counterexample with $n = 9$.

Problem 10.2: Consider the following search problem:

MAX-RECTANGLE-COVERAGE:
Input: a set R of m rectangles, a set P of n points in 2D, and a number $k \leq m$.
Output: Find a subset $Q \subseteq R$ of k rectangles, while maximizing the number of points of P covered by Q.

(Here, a point $p \in P$ is *covered* by Q iff p is contained in r for some rectangle $r \in Q$.)

Consider the following decision problem:

MAX-RECTANGLE-COVERAGE-DECISION:
Input: a set R of m rectangles, a set P of n points in 2D, and numbers $k \leq m$ and $\ell \leq n$.
Output: True iff there exists a subset $Q \subseteq R$ of k rectangles, such that the number of points of P covered by Q is at least ℓ.

Prove that MAX-RECTANGLE-COVERAGE has a polynomial-time algorithm iff MAX-RECTANGLE-COVERAGE-DECISION has a polynomial-time algorithm.

(Note: one direction should be easy; for the other direction, see lab 12b for examples of this type of question.)