Problem 10.1: Consider the following problem: given a set $S$ of $n$ distinct positive integers and a number $L$, we want to form the largest number of disjoint pairs in $S$, such that the sum of each pair is at most $L$.

For example, for $S = \{1, 3, 4, 5, 8\}$ and $L = 10$, an optimal solution has 2 pairs (e.g., one optimal solution is $\{1, 8\}$ and $\{4, 5\}$, but there are a number of other solutions that are also optimal).

Consider the following greedy strategy:

```
repeat
   find a pair \{a, b\} \subseteq S with a + b \leq L maximizing a + b
   if no such pair exists then stop
   else output the pair \{a, b\}, and remove a and b from S
```

(a) (85 pts) Prove that this algorithm always correctly finds an optimal solution.

Like various greedy correctness proofs from class, you should use an “exchange” argument: let $\{a, b\} \subseteq S$ with $a + b \in L$ maximizing $a + b$; suppose $P^*$ is an optimal solution, and suppose $\{a, b\}$ is not in $P^*$; show how to modify $P^*$ to get a different feasible solution that is just as good and that uses $\{a, b\}$...

(b) (15 pts) Consider a variant of the problem, where we want to find the largest number of disjoint triples, such that the sum of each triple is at most $L$. Show that the same greedy strategy (where now we find a triple $\{a, b, c\} \subseteq S$ with $a + b + c \leq L$ maximizing $a + b + c$ at each iteration) does not always correctly find an optimal solution. You should give a counterexample with $n = 9$.

Problem 10.2: Consider the following search problem:

**MAX-RECTANGLE-COVERAGE:**

- **Input:** a set $R$ of $m$ rectangles, a set $P$ of $n$ points in 2D, and a number $k \leq m$.
- **Output:** Find a subset $Q \subseteq R$ of $k$ rectangles, while maximizing the number of points of $P$ covered by $Q$.

(Here, a point $p \in P$ is covered by $Q$ iff $p$ is contained in $r$ for some rectangle $r \in Q$.)

Consider the following decision problem:

**MAX-RECTANGLE-COVERAGE-DECISION:**

- **Input:** a set $R$ of $m$ rectangles, a set $P$ of $n$ points in 2D, and numbers $k \leq m$ and $\ell \leq n$.
- **Output:** True iff there exists a subset $Q \subseteq R$ of $k$ rectangles, such that the number of points of $P$ covered by $Q$ is at least $\ell$. 

Prove that MAX-RECTANGLE-COVERAGE has a polynomial-time algorithm iff MAX-RECTANGLE-COVERAGE-DECISION has a polynomial-time algorithm.

(Note: one direction should be easy; for the other direction, see lab 12b for examples of this type of question.)