CS/ECE 374 A (Spring 2024)
Homework 1 (due Jan 25 Thursday at 10am)

Instructions: Carefully read `https://courses.grainger.illinois.edu/cs374al1/sp2024/hw_policies.html` and `https://courses.grainger.illinois.edu/cs374al1/sp2024/integrity.html`.

- **Groups of up to three people may submit joint solutions.** Each problem should be submitted by exactly one person, and the beginning of the homework should clearly state the Gradescope names and email addresses of each group member. In addition, whoever submits the homework must tell Gradescope who their other group members are.
- **Submit your solutions electronically on the course Gradescope site as PDF files.** Submit a separate PDF file for each numbered problem. If you plan to typeset your solutions, please use the LaTeX solution template on the course web site. If you must submit scanned handwritten solutions, please use a black pen on blank white paper and a high-quality scanner app (or an actual scanner, not just a phone camera).
- If you are not using your real name and your `illinois.edu` email address on Gradescope, you will need to fill in the form linked in the course web page.
- **You may use any source at your disposal**—paper, electronic, or human—but you *must* cite *every* source that you use, and you *must* write everything yourself in your own words.
- Any homework or exam solution that breaks any of the following rules could be given a *zero*.
  - Always give complete solutions, not just examples.
  - Always declare all your variables, in English. In particular, always describe the specific problem your algorithm is supposed to solve.
  - Always avoid unnecessarily long answers.

Problem 1.1: Let $L \subseteq \{0,1\}^*$ be the language defined recursively as follows:

- The empty string ε is in L.
- For any string x in L, the strings $1x01$ and $11x0$ are also in L.
- For any strings x, y such that xy is in L, the string $x011y$ is also in L. (In other words, inserting 011 anywhere to a string in L yields another string in L.)
- The only strings in L are those that can be obtained by the above rules.

Define $L_{\text{double}} = \{x \in \{0,1\}^* : \#_1(x) = 2\#_0(x)\}$, where $\#_a(x)$ denotes the number of occurrences of the symbol a in the string x.

(a) Prove that $L \subseteq L_{\text{double}}$, by using induction. (You should use *strong* induction.)

(b) Conversely, prove that $L_{\text{double}} \subseteq L$, by using induction.

[Hint: What does a string that does not contain 011 as a substring look like? It may be helpful to decompose the string into blocks of 0's and 1's (i.e., write it as $0^{i_1}1^{j_1}0^{i_2}1^{j_2}\cdots 0^{i_k}1^{j_k}$ with $i_1, j_1, \ldots, i_k, j_k \geq 1$, if it starts with a 0 and ends with a 1; and similarly for the other cases).]

Problem 1.2: Consider the following recurrence:

$$T(n) \;=\; \begin{cases} 2\,T(\lfloor 2n/3 \rfloor) + 11\,T(\lfloor n/3 \rfloor) + 2024n + 374 & \text{if } n \geq 3 \\ 5 & \text{if } n = 1 \text{ or } n = 2 \end{cases}$$

Prove that $T(n) = O(n^3)$ by induction.

[Hint: try an induction hypothesis of the form $T(n) \leq cn^3 - c'n - c''$ for some constants c, c', c'' to be determined.]