Proving Non-regularity

Lecture 6

February 2, 2023

Regular Languages, DFAs, NFAs

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- Hence number of regular languages is *countably infinite*
- Number of languages is *uncountably infinite*
- Hence there must be a non-regular language!

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How do we formalize intuition and come up with a formal proof?

Proof by Contradiction

- Suppose L is regular. Then there is a DFA M such that $L(M) = L$.
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What is the behavior of M on these strings? Let $q_i = \delta^*(s, 0^i)$.

By pigeon hole principle $q_i = q_j$ for some $0 \le i < j \le n$.
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M should accept $0^i 1^i$ but then it will also accept $0^j 1^i$ where $i \neq j$.
This contradicts the fact that M accepts L. Thus, there is no DFA for L.

Generalizing the argument

Definition

For a language L over Σ and two strings $x, y \in \Sigma^*$ we say that x and y are distinguishable with respect to L if there is a string $w \in \Sigma^*$ such that exactly one of xw, yw is in L. In other words either $xw \in L, yw \notin L$ or $xw \notin L, yw \in L$.

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Important: Indistiguishability is with respect to a specific language

Wee Lemma

Lemma

Suppose $L = L(M)$ *for some DFA* $M = (Q, \Sigma, \delta, s, A)$ *and suppose* x, y *are distinguishable with respect to* L*. Then* $\delta^*(s, x) \neq \delta^*(s, y)$*.*

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Proof.

Since x, y are distinguishable let w be the distinguishing suffix. If $\delta^*(s, x) = \delta^*(s, y)$ then M will either accept both the strings xw, yw, or reject both. But exactly one of them is in L, a contradiction. □

Fooling Sets

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Suppose there is a DFA $M = (Q, \Sigma, \delta, s, A)$ that accepts L. Let $|Q| = n$.
If $n < |F|$ then by pigeon hole principle there are two strings $x, y \in F$, $x \neq y$ such that $\delta^*(s, x) = \delta^*(s, y)$ but x, y are distinguishable.

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Implies that there is w such that exaclty one of xw, yw is in L.
However, M's behaviour on xw and yw is exacly the same and hence M will accept both xw, yw or reject both. A contradiction. □

Infinite Fooling Sets

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Corollary

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Proof.

Suppose for contradiction that $L = L(M)$ for some DFA M with n states.
Any subset F' of F is a fooling set. (Why?) Pick $F' \subseteq F$ arbitrarily such that $|F'| > n$. By preceding theorem, we obtain a contradiction. □

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- $\{0^{k^2} \mid k \geq 0\}$
- $\{w \mid w$ is valid Python program$\}$

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Claim

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Why?

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Why?

- Suppose $a_1 a_2 \ldots a_k$ and $b_1 b_2 \ldots b_k$ are two distinct bitstrings of length k
- Let i be first index where $a_i \neq b_i$
- $y = 0^{i-1}$ is a distinguishing suffix for the two strings

How to pick a fooling set

How do we pick a fooling set F?

- If x, y are in F and $x \neq y$ they should be distinguishable! Of course.
- All strings in F except maybe one should be prefixes of strings in the language L.
  For example if $L = \{0^k 1^k \mid k \geq 0\}$ do not pick 1 and 10 (say). Why?
- Instead of picking an infinite fooling set one can also show that for every $n > 0$ there is a fooling set F_n such that $|F_n| \geq n$. See Chandra's notes on the webpage for why this may help in several cases.

Part I

Non-regularity via closure properties

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$L' = \{0^k 1^k \mid k \geq 0\}$

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$L' = L \cap L(0^*1^*)$

Claim: The above and the fact that L' is non-regular implies L is non-regular. Why?

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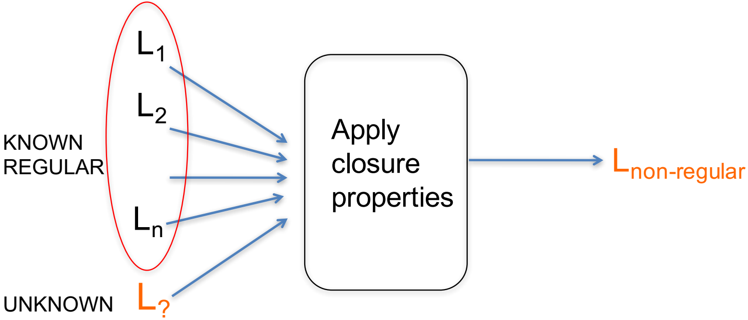
Claim: The above and the fact that L' is non-regular implies L is non-regular. Why?

Suppose L is regular. Then since $L(0^*1^*)$ is regular, and regular languages are closed under intersection, L' also would be regular. But we know L' is not regular, a contradiction.

Non-regularity via closure properties

General recipe:



Proving non-regularity: Summary

- DFAs have fixed memory. Any language that requires memory that grows with input size is not regular. Not always easy to tell!
- Method of distinguishing suffixes. To prove that L is non-regular find an infinite fooling set.
- Closure properties. Use existing non-regular languages and regular languages to prove that some new language is non-regular.
- Pumping lemma. We did not cover it but it is sometimes an easier proof technique to apply, but not as general as the fooling set technique.

Part II

Myhill-Nerode Theorem (Optional)

Indistinguishability

Recall:

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Given language L over Σ define a relation $\equiv_L$ over strings in Σ^* as follows: $x \equiv_L y$ iff x and y are indistinguishable with respect to L.

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$\equiv_L$ is an equivalence relation over Σ^*.

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Claim

Let x, y be two distinct strings. If x, y belong to the same equivalence class of $\equiv_L$ then x, y are indistinguishable. Otherwise they are distinguishable.

Corollary

If $\equiv_L$ is finite with n equivalence classes then there is a fooling set F of size n for L. If $\equiv_L$ is infinite then there is an infinite fooling set for L.

Myhill-Nerode Theorem

Theorem (Myhill-Nerode)

L is is regular if and only if $\equiv_L$ has a finite number of equivalence classes. If $\equiv_L$ is finite with n equivalence classes then there is a DFA M accepting L with exactly n states and this is the minimum possible.

Corollary

A language L is non-regular if and only if there is an infinite fooling set F for L.

Algorithmic implication: For every DFA M one can find in polynomial time a DFA M' such that $L(M) = L(M')$ and M' has the fewest possible states among all such DFAs.