Let L be an arbitrary regular language over the alphabet $\Sigma = \{\texttt{0}, \texttt{1}\}$. Prove that the following languages are also regular. (You probably won't get to all of these.)

1. $\textsc{FlipOdds}(L) := \{\mathit{flipOdds}(w) \mid w \in L\}$, where the function *flipOdds* inverts every odd-indexed bit in w. For example:

$$\mathit{flipOdds}(\texttt{0000111101010101}) = \texttt{1010010111111111}$$

2. $\textsc{UnflipOdd1s}(L) := \{w \in \Sigma^* \mid \mathit{flipOdd1s}(w) \in L\}$, where the function *flipOdd1* inverts every other 1 bit of its input string, starting with the first 1. For example:

$$\mathit{flipOdd1s}(\texttt{0000\underline{1}1\underline{1}10\underline{1}010\underline{1}01}) = \texttt{0000\underline{0}1\underline{0}10\underline{0}010\underline{0}01}$$

3. FLIPODD1S(L) := $\{$*flipOdd1s*$(w) \mid w \in L\}$, where the function *flipOdd1* is defined as in the previous problem.

4. Prove that the language *insert*1$(L) := \{x\texttt{1}y \mid xy \in L\}$ is regular.

   Intuitively, *insert*1(L) is the set of all strings that can be obtained from strings in L by inserting exactly one `1`. For example, if $L = \{\varepsilon, \texttt{OOK!}\}$, then *insert*1$(L) = \{\texttt{1}, \texttt{1OOK!}, \texttt{O1OK!}, \texttt{OO1K!}, \texttt{OOK1!}, \texttt{OOK!1}\}$.

Work on these later:

5. Prove that the language $delete\mathtt{1}(L) := \{xy \mid x\mathtt{1}y \in L\}$ is regular.

   Intuitively, $delete\mathtt{1}(L)$ is the set of all strings that can be obtained from strings in L by deleting exactly one $\mathtt{1}$. For example, if $L = \{\mathtt{101101}, \mathtt{00}, \varepsilon\}$, then $delete\mathtt{1}(L) = \{\mathtt{01101}, \mathtt{10101}, \mathtt{10110}\}$.

6. Consider the following recursively defined function on strings:

$$stutter(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ aa \bullet stutter(x) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

   Intuitively, $stutter(w)$ doubles every symbol in w. For example:

   - $stutter(\mathtt{PRESTO}) = \mathtt{PPRREESSTTOO}$
   - $stutter(\mathtt{HOCUS\diamond POCUS}) = \mathtt{HHOOCCUUSS\diamond\diamond PPOOCCUUSS}$

   (a) Prove that the language $stutter^{-1}(L) := \{w \mid stutter(w) \in L\}$ is regular.

   (b) Prove that the language $stutter(L) := \{stutter(w) \mid w \in L\}$ is regular.

7. Consider the following recursively defined function on strings:

$$evens(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ \varepsilon & \text{if } w = a \text{ for some symbol } a \\ b \cdot evens(x) & \text{if } w = abx \text{ for some symbols } a \text{ and } b \text{ and some string } x \end{cases}$$

   Intuitively, $evens(w)$ skips over every other symbol in w. For example:

   - $evens(\mathtt{EXPELLIARMUS}) = \mathtt{XELAMS}$
   - $evens(\mathtt{AVADA\diamond KEDAVRA}) = \mathtt{VD\diamond EAR}$.

   (a) Prove that the language $evens^{-1}(L) := \{w \mid evens(w) \in L\}$ is regular.

   (b) Prove that the language $evens(L) := \{evens(w) \mid w \in L\}$ is regular.

You may find it helpful to imagine these transformations concretely on the following DFA for the language specified by the regular expression $\texttt{00}^*\texttt{11}^*$.