

Here are the formal recursive definitions of string length, concatenation, and reversal:

$$|w| := \begin{cases} 0 & \text{if } w = \varepsilon \\ 1 + |x| & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

$$w \cdot z := \begin{cases} z & \text{if } w = \varepsilon \\ a \cdot (x \cdot z) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

$$w^R := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ x^R \cdot a & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

Lemma 1: $w \cdot \varepsilon = w$ for all strings w .

Lemma 2: $|w \cdot z| = |w| + |z|$ for all strings w and z .

Lemma 3: $(w \cdot y) \cdot z = w \cdot (y \cdot z)$ for all strings w , y , and z .

1. Prove that $|w^R| = |w|$ for every string w .

Solution (induction on w):

Let w be an arbitrary string.

Assume for every string x where $|x| < |w|$ that $|x^R| = |x|$.

There are two cases to consider.

- If $w = \varepsilon$, then

$$\begin{aligned} |w^R| &= |\varepsilon^R| && \text{because } w = \varepsilon \\ &= |\varepsilon| && \text{by definition of } ^R \\ &= |w| && \text{because } w = \varepsilon \end{aligned}$$

- Otherwise, $w = ax$ for some symbol a and some string x , and therefore

$$\begin{aligned} |w^R| &= |(ax)^R| && \text{because } w = ax \\ &= |x^R \cdot a| && \text{by definition of } ^R \\ &= |x^R| + |a| && \text{by Lemma 2} \\ &= |x^R| + 1 + |\varepsilon| && \text{by definition of } |\cdot| \\ &= |x^R| + 1 && \text{by definition of } |\cdot| \\ &= |x| + 1 && \text{by the induction hypothesis} \\ &= |ax| && \text{by definition of } |\cdot| \\ &= |w| && \text{because } w = ax \end{aligned}$$

In both cases, we conclude that $|w^R| = |w|$. ■

2. Prove that $(w \cdot z)^R = z^R \cdot w^R$ for all strings w and z .

Solution (induction on w):

Let w and z be arbitrary strings.

Assume for every string x where $|x| < |w|$ that $(x \cdot z)^R = z^R \cdot x^R$.

There are two cases to consider:

- If $w = \varepsilon$, then

$$\begin{aligned}
 (w \cdot z)^R &= (\varepsilon \cdot z)^R && \text{because } w = \varepsilon \\
 &= z^R && \text{by definition of } \cdot \\
 &= z^R \cdot \varepsilon && \text{by Lemma 1} \\
 &= z^R \cdot \varepsilon^R && \text{by definition of } ^R \\
 &= z^R \cdot w^R && \text{because } w = \varepsilon
 \end{aligned}$$

- Otherwise, $w = ax$ for some symbol a and some string x .

$$\begin{aligned}
 (w \cdot z)^R &= (ax \cdot z)^R && \text{because } w = ax \\
 &= (a \cdot (x \cdot z))^R && \text{by definition of } \cdot \\
 &= (x \cdot z)^R \cdot a && \text{by definition of } ^R \\
 &= (z^R \cdot x^R) \cdot a && \text{by the induction hypothesis } (*) \\
 &= z^R \cdot (x^R \cdot a) && \text{by Lemma 3} \\
 &= z^R \cdot (ax)^R && \text{by definition of } ^R \\
 &= z^R \cdot w^R && \text{because } w = ax
 \end{aligned}$$

In both cases, we conclude that $(w \cdot z)^R = z^R \cdot w^R$. ■

*How did I know that the induction hypothesis needs to change the first string w , but not the second string z ? I actually wrote down the inductive **argument** first, and then **noticed** that I needed to argue inductively about $x \cdot z$ at line $(*)$. Same string z , but w changed to x .*

Alternatively, we could notice that the recursive definition of $w \cdot z$ recurses on w but leaves z unchanged. Inductive proofs always mirror the recursive definitions of the objects in question.

*Alternatively, in light of Lemma 2, we could have used induction on the **sum** of the string lengths. Then the inductive hypothesis would read “Assume for all strings x and y such that $|x| + |y| < |w| + |z|$ that $(x \cdot y)^R = x^R \cdot y^R$.”*

3. Prove that $(w^R)^R = w$ for every string w .

Solution (induction on w):

Let w be an arbitrary string.

Assume for every string x where $|x| < |w|$ that $(x^R)^R = x$.

There are two cases to consider.

- If $w = \varepsilon$, then

$$\begin{aligned}
 (w^R)^R &= (\varepsilon^R)^R && \text{because } w = \varepsilon \\
 &= \varepsilon^R && \text{by definition of } ^R \\
 &= \varepsilon && \text{by definition of } ^R \\
 &= w && \text{because } w = \varepsilon
 \end{aligned}$$

- Otherwise, $w = ax$ for some symbol a and some string x .

$$\begin{aligned}
 (w^R)^R &= ((ax)^R)^R && \text{because } w = ax \\
 &= (x^R \cdot a)^R && \text{by definition of } ^R \\
 &= a^R \cdot (x^R)^R && \text{by problem 2} \\
 &= (\varepsilon^R \cdot a) \cdot (x^R)^R && \text{by definition of } ^R \\
 &= (\varepsilon \cdot a) \cdot (x^R)^R && \text{by definition of } ^R \\
 &= a \cdot (x^R)^R && \text{by definition of } \cdot \\
 &= a \cdot (x^R)^R && \text{by definition of } \cdot \\
 &= a \cdot x && \text{by the induction hypothesis} \\
 &= w && \text{because } w = ax
 \end{aligned}$$

In both cases, we conclude that $(w^R)^R = w$. ■

To think about later: Let $\#(a, w)$ denote the number of times symbol a appears in string w . For example, $\#(\text{X}, \text{WTF374}) = 0$ and $\#(0, 000010101010010100) = 12$.

4. Give a formal recursive definition of $\#(a, w)$.

Solution:

$$\#(a, w) = \begin{cases} 0 & \text{if } w = \varepsilon \\ 1 + \#(a, x) & \text{if } w = ax \text{ for some string } x \\ \#(a, x) & \text{if } w = bx \text{ for some symbol } b \neq a \text{ and some string } x \end{cases}$$

■

Solution (clever notation):

$$\#(a, w) = \begin{cases} 0 & \text{if } w = \varepsilon \\ \#(a, x) + [a = b] & \text{if } w = bx \text{ for some symbol } b \text{ and some string } x \end{cases}$$

The expression $[a = b]$ in red is [Iverson bracket notation](#). For any proposition P , the expression $[P]$ is equal to 1 if P is true and 0 if P is false. ■

5. Prove that $\#(a, w \cdot z) = \#(a, w) + \#(a, z)$ for all symbols a and all strings w and z .

Solution (induction on w):

Let a be an arbitrary symbol, and let w and z be arbitrary strings.

Assume for any string x such that $|x| < |w|$ that $\#(a, x \cdot z) = \#(a, x) + \#(a, z)$.

There are three cases to consider.

- If $w = \varepsilon$, then

$$\begin{aligned} \#(a, w \cdot z) &= \#(a, \varepsilon \cdot z) && \text{because } w = \varepsilon \\ &= \#(a, z) && \text{by definition of } \cdot \\ &= \#(a, \varepsilon) + \#(a, z) && \text{by definition of } \# \\ &= \#(a, w) + \#(a, z) && \text{because } w = \varepsilon \end{aligned}$$

- If $w = ax$ for some string x , then

$$\begin{aligned} \#(a, w \cdot z) &= \#(a, ax \cdot z) && \text{because } w = ax \\ &= \#(a, a \cdot (x \cdot z)) && \text{by definition of } \cdot \\ &= 1 + \#(a, x \cdot z) && \text{by definition of } \# \\ &= 1 + \#(a, x) + \#(a, z) && \text{by the induction hypothesis} \\ &= \#(a, ax) + \#(a, z) && \text{by definition of } \# \\ &= \#(a, w) + \#(a, z) && \text{because } w = ax \end{aligned}$$

- If $w = bx$ for some symbol $b \neq a$ and some string x , then

$$\begin{aligned} \#(a, w \cdot z) &= \#(a, bx \cdot z) && \text{because } w = bx \\ &= \#(a, b \cdot (x \cdot z)) && \text{by definition of } \cdot \\ &= \#(a, x \cdot z) && \text{by definition of } \# \\ &= \#(a, x) + \#(a, z) && \text{by the induction hypothesis} \\ &= \#(a, bx) + \#(a, z) && \text{by definition of } \# \\ &= \#(a, w) + \#(a, z) && \text{because } w = bx \end{aligned}$$

In every case, we conclude that $\#(a, w \cdot z) = \#(a, w) + \#(a, z)$. ■

6. Prove that $\#(a, w^R) = \#(a, w)$ for all symbols a and all strings w .

Solution (induction on w): Let a be an arbitrary symbol, and let w be an arbitrary string.

Assume for any string x such that $|x| < |w|$ that $\#(a, x^R) = \#(a, x)$.

There are three cases to consider.

- If $w = \varepsilon$, then $w^R = \varepsilon = w$ by definition, so $\#(a, w^R) = \#(a, w)$.
- If $w = ax$ for some string x , then

$$\begin{aligned}
 \#(a, w^R) &= \#(a, (ax)^R) && \text{because } w = ax \\
 &= \#(a, x^R \cdot a) && \text{by definition of } ^R \\
 &= \#(a, x^R) + \#(a, a) && \text{by problem 5} \\
 &= \#(a, x^R) + 1 && \text{by definition of } \# \\
 &= \#(a, x) + 1 && \text{by the induction hypothesis} \\
 &= \#(a, ax) && \text{by definition of } \# \\
 &= \#(a, w) && \text{because } w = ax
 \end{aligned}$$

- If $w = bx$ for some symbol $b \neq a$ and some string x , then

$$\begin{aligned}
 \#(a, w^R) &= \#(a, (bx)^R) && \text{because } w = bx \\
 &= \#(a, x^R \cdot b) && \text{by definition of } ^R \\
 &= \#(a, x^R) + \#(a, b) && \text{by problem 5} \\
 &= \#(a, x^R) && \text{by definition of } \# \\
 &= \#(a, x) && \text{by the induction hypothesis} \\
 &= \#(a, bx) && \text{by definition of } \# \\
 &= \#(a, w) && \text{because } w = bx
 \end{aligned}$$

In every case, we conclude that $\#(a, w^R) = \#(a, w)$. ■