1. FIGURE EIGHT: A ***figure eight of size ℓ*** is an undirected graph with $2\ell-1$ vertices consisting of two cycles C_1 and C_2 of ℓ vertices each sharing exactly one vertex. Prove that it is NP-hard to decide, given an undirected graph $G=(V,E)$ and an integer ℓ, whether G has a figure eight of size at least ℓ as a subgraph.

Solution: We prove that FIGURE EIGHT is NP-hard by reduction from the standard UNDIRHAMCYCLE problem.

Let $G=(V,E)$ be an arbitrary undirected graph, and let $n=|V|$. We construct a new graph $H=(V',E')$ by taking two disjoint copies of G and identifying the two copies of an arbitrary vertex u. More formally:

- $V' := ((V \setminus \{u\}) \times \{1,2\}) \cup \{u\}$
- $E' := \{u(v,z) \mid v \in (V \setminus \{u\}), z \in \{1,2\}\}$
  $\cup \{(v,z)(w,z) \mid v,w \in (V \setminus \{u\}), z \in \{1,2\}\}$

Because H has $2n-1$ vertices, it cannot contain a figure eight of size larger than n. I claim that G contains a Hamiltonian cycle if and only H contains a figure eight of size n.

$\Longrightarrow$ Suppose G contains a Hamiltonian cycle C. Let F be the subgraph of H consisting of both copies of C sharing the vertex u. This subgraph F is a figure eight of size n.

$\Longleftarrow$ Suppose H contains a figure eight F of size n. The only connection between the two copies of G is the vertex u, so a cycle of length n can only exist entirely within one copy of G or the other. Further, such a cycle must contain every vertex of that copy of G exactly once. Therefore, either cycle C of F is a Hamiltonian cycle in G.

Given G, we can construct H in polynomial time by brute force. ■

2. HIGH-DEGREE INDEPENDENT SET: Suppose we are given a graph G and an integer k. Prove that it is NP-hard to decide whether G contains an independent set of k vertices, each of which has degree at least k in G.

Solution: We prove that HIGH-DEGREE INDEPENDENT SET is NP-hard by reduction from the standard INDEPENDENT SET problem.

Suppose we are given an arbitrary graph G and an arbitrary integer $k > 1$. (When $k = 1$, the INDEPENDENT SET problem is trivial!) To determine whether G contains an independent set of size k, we construct a new graph H by attaching a "fan" of k edges to each vertex of G.

I claim that G contains an independent set of size k if and only if H contains an independent set of size k, each of whose vertices has degree at least k.

$\Longrightarrow$ Suppose G contains an independent set S of size k. Every vertex of S has degree at least k in H.

$\Longleftarrow$ Suppose H contains an independent set S of size k, each of whose vertices has degree at least k in H. The vertices at the ends of the added fans each have degree 1, so they cannot be in S. Thus, S is a subset of vertices in G. Each edge of G is also an edge of H, and therefore contains at most one vertex in S. We conclude that S is an independent set of size k in G.

Given G, we can construct H in polynomial time by brute force. ■

3. HALF-CLIQUE: Suppose we are given a graph G with $2n$ vertices, for some integer n. Prove that it is NP-hard to decide whether G contains a complete subgraph with n vertices.

Solution: We prove that HALF-CLIQUE is NP-hard by reduction from the standard CLIQUE problem.

Suppose we are given an arbitrary graph $G = (V, E)$ and an arbitrary positive integer k. Suppose G has n vertices. We construct a new graph H by adding n new vertices, of which $n-k$ are connected to each other and to every vertex of G, and the other k have no edges at all. More formally, we have $H = (V', E')$ where

- $V' = V \cup X \cup Z$, where X is a set of $n-k$ new vertices, and Z is another set of k new vertices.
- $E' = E \cup \{uv \mid u \in X \text{ and } v \in V \cup X\}$

In total, H has exactly $2n$ vertices. I claim that G contains a clique of size k if and only if H contains a clique of size n.

$\Longrightarrow$ Suppose G contains a clique K of size k. Then $K \cup X$ is a clique of size $k + (n-k) = n$ in H.

$\Longleftarrow$ Suppose H contains a clique K of size n. If K omits any vertex $x \in X$, adding x and removing some other vertex in $K \setminus X$ gives us another clique of size n that contains more vertices of X. It follows by induction that H contains a clique K' of size n that includes *every* vertex in X. Then $K' \setminus X$ is a clique of size k in G.

We can construct H from G in polynomial time by brute force. ■

Solution (three cases): We prove that HALF-CLIQUE is NP-hard by reduction from the standard CLIQUE problem.

Suppose we are given an arbitrary graph $G = (V, E)$ and an arbitrary positive integer k. Suppose G has n vertices. There are three cases to consider.

- **Suppose $n > 2k$.** We construct a new graph H by adding a clique of size $n-2k$ to G, along with edges between that clique and every vertex of G. More formally, we have $H = (V', E')$ where
  - $V' = V \cup Z$ where $Z = \{z_i \mid 1 \le i \le n-2k\}$
  - $E' = E \cup \{wz_i \mid w \in V'\}$

  In total, H has $2n-2k$ vertices. I claim that G contains a clique of size k if and only if H contains a clique of size $n-k$.

  $\Longrightarrow$ Suppose G has a clique K of size k. Then $K \cup Z$ is a clique of size $k + (n-2k) = n-k$ in H.

  $\Longleftarrow$ Suppose H has a clique K of size $n-k$. Then $K \setminus Z$ is a clique of size *at least* $(n-k) - (n-2k) = k$ in G.

- **Suppose $n = 2k$.** In this case the reduction is trivial: Let $H = G$. Then G contains a clique of size k if and only if H contains a clique of size $n/2$.

- **Suppose $n < 2k$.** We construct a new graph H by adding $2k - n$ isolated vertices to G. In total, H has $2k$ vertices. I claim that G contains a clique of size k if and only if H contains a clique of size k.

  $\Longrightarrow$ Suppose G has a clique K of size k. Then K is also a clique of size k in H.

  $\Longleftarrow$ Suppose H has a clique K of size k. None of the $2k - n$ isolated vertices of H are in K, so in fact K is a clique of size k in G.

In all three cases, G contains a clique of size k if and only if half of the vertices of H form a clique. Moreover, in all three cases, we can construct H from G in polynomial time by brute force. ■