1. (a) Give a short argument that PERFECTMATCHING $\in$ NP.

Solution (direct): We can use a perfect matching $M \subseteq E$ as a certificate. The verifier will make sure every vertex is incident to exactly one edge in M. ■

Solution (Cook-Levin): We can use a polynomial size string $y \in \{0,1\}^*$ representing $|y|$ Booleans as a certificate. The verifier will perform the reduction in part (b) to create a Boolean formula Φ and then check if the Booleans given by y satisfy Φ. ■

Rubric: 1 point.

(b) Describe a polynomial time many-one reduction *from* PERFECTMATCHING *to* SAT (yes, you have to describe it yourself; do not merely defer to the Cook-Levin theorem).

Solution: Let $G = (V, E)$ be an input to PERFECTMATCHING. We'll assume G is simple. We build a CNF Boolean formula Φ as follows.

If G contains an isolated vertex with no incident edges, then there is no perfect matching. We build the unsatisfiable formula $z \wedge \bar{z}$ in $O(|V| + |E|)$ time. We'll assume from here on that G has no isolated vertices.

Formula Φ contains a variable x_e for each edge $e \in E$. For each vertex u, formula Φ contains the clause

$$\bigvee_{uv \in E} x_{uv}.$$

Finally, for each pair of adjacent edges $uv, uw \in E$, formula Φ contains the clause

$$(\overline{x_{uv}} \vee \overline{x_{uw}}).$$

We need to prove

G has a perfect matching
if and only if
Φ is satisfiable.

$\Longrightarrow$ Suppose G has a perfect matching $M \subseteq E$. Consider an assignment to Φ's variables where $x_e = \text{TRUE}$ if and only if $e \in M$. For each vertex u, at least one edge $uw \in M$, implying the clause $\bigvee_{uv \in E} x_{uv}$ contains the true literal x_{uw}. Also, for each pair of adjacent edges $uv, uw \in E$, it's the case that $uv \notin M$ or $uw \notin M$, implying the clause $(\overline{x_{uv}} \vee \overline{x_{uw}})$ contains a true literal.

$\Longleftarrow$ Suppose Φ is satisfiable by some assignment to the variables. Let $M \subseteq E$ be the set of edges where x_e is assigned TRUE. For each vertex u, the clause $\bigvee_{uv \in E} x_{uv}$ contains a true literal, implying u is incident to at least one edge $uw \in M$. Also, for each pair of adjacent edges $uv, uw \in E$, the clause $(\overline{x_{uv}} \vee \overline{x_{uw}})$ contains a true literal, implying $uv \notin M$ or $uw \notin M$.

We can easily construct Φ in polynomial $O(|V| + |E|)$ time. ■

Rubric: 8 points: standard *NP-hardness* rubric without the part about choosing a problem (scaled). No penalty for assuming every vertex has at least one incident edge.

(c) There exists a polynomial time algorithm for PERFECTMATCHING (it's beyond the scope of this course). Why doesn't that algorithm together with the result from part (b) imply P = NP?

Solution: For the polynomial time algorithm to prove P = NP, we would need to reduce *from* SAT *to* PERFECTMATCHING, but part (b) asks for the opposite! ■

Rubric: 1 point.

2. ***Prove*** that the following problem (which we call StringMatch) is NP-hard. The input is a finite set S of strings, all of the same length n, over the alphabet $\{0, 1, 2\}$. The problem is to determine whether there is a string $w \in \{0, 1\}^n$ such that for every string $s \in S$, the strings s and w have the same symbol in at least one position.

Solution: We'll do a reduction from 3SAT. Let Φ be a 3CNF Boolean formula with n variables $\{x_1, \ldots, x_n\}$ and k clauses. We'll build an instance S of StringMatch as follows. For each clause C in Φ, set S contains a string s_C where the ith symbol is set to

- 0 if literal $\overline{x_i}$ is in C,
- 1 if literal x_i is in C, or
- 2 otherwise.

We need to prove

Φ is satisfiable if and only if there is a string w satisfying the StringMatch condition.

$\Longrightarrow$ Suppose Φ has a satisfying assignment of its variables. Let w be the string where the ith symbol is 0 if x_i is assigned False and 1 otherwise. For each clause C, the assignment contains a true literal. If the true literal is some $\overline{x_i}$, then the ith symbol of both s_C and w is 0. If the true literal is some x_i, then the ith symbol of both s_C and w is 1. String w satisfies the StringMatch condition.

$\Longleftarrow$ Suppose there is a string w that satisfies the StringMatch condition. Consider an assignment to Φ's variables where x_i is assigned False if the ith symbol of w is 0 and True otherwise. For each clause C, let i be such that s_C and w agree on the ith symbol. If that symbol is 0, then variable x_i is assigned False and s_C contains the true literal $\overline{x_i}$. Otherwise, variable x_i is assigned True and s_C contains the true literal x_i. The assignment satisfies Φ.

We can easily construct S in polynomial $O(kn)$ time. ■

Rubric: 10 points: standard NP-hardness rubric.