1. Let $a \in \{\mathtt{0}, \mathtt{1}\}$. The ***flip*** of a, denoted $\bar{a}$ is defined as

$$\bar{a} := \begin{cases} \mathtt{1} & \text{if } a = \mathtt{0} \\ \mathtt{0} & \text{if } a = \mathtt{1} \end{cases}$$

Consider the following pair of mutually recursive functions on strings over $\{\mathtt{0}, \mathtt{1}\}$.

$$\mathit{flipodds}(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ \bar{a} \cdot \mathit{flipevens}(x) & \text{if } w = ax \end{cases} \qquad \mathit{flipevens}(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ a \cdot \mathit{flipodds}(x) & \text{if } w = ax \end{cases}$$

(a) Prove for all strings $w \in \{\mathtt{0}, \mathtt{1}\}^*$, we have

$$(\mathit{flipodds}(w))^c = \mathit{flipevens}(w).$$

Solution: Let w be an arbitrary string.

Assume $(\mathit{flipodds}(x))^c = \mathit{flipevens}(x)$ for every string $x \in \{\mathtt{0}, \mathtt{1}\}^*$ such that $|x| < |w|$.

There are two cases to consider (mirroring the definition of *flipodds*):

- If $w = \varepsilon$, then

$$\begin{aligned} (\mathit{flipodds}(w))^c &= (\mathit{flipodds}(\varepsilon))^c && \text{because } w = \varepsilon \\ &= (\varepsilon)^c && \text{by definition of } \mathit{flipodds} \\ &= \varepsilon && \text{by definition of } (\cdot)^c \\ &= \mathit{flipevens}(\varepsilon) && \text{by definition of } \mathit{flipevens} \\ &= \mathit{flipevens}(w) && \text{because } w = \varepsilon \end{aligned}$$

- If $w = ax$ for some $a \in \{\mathtt{0}, \mathtt{1}\}$ and $x \in \{\mathtt{0}, \mathtt{1}\}^*$, then

$$\begin{aligned} (\mathit{flipodds}(w))^c &= (\mathit{flipodds}(ax))^c && \text{because } w = ax \\ &= (\bar{a} \cdot \mathit{flipevens}(x))^c && \text{by definition of } \mathit{flipodds} \\ &= a \cdot (\mathit{flipevens}(x))^c && \text{by definition of } (\cdot)^c \\ &= a \cdot ((\mathit{flipodds}(x))^c)^c && \text{by the inductive hypothesis} \\ &= a \cdot \mathit{flipodds}(x) && \text{because complement is an involution} \\ &= \mathit{flipevens}(ax) && \text{by definition of } \mathit{flipevens} \\ &= \mathit{flipevens}(w) && \text{because } w = ax \end{aligned}$$

In both cases, we conclude that $(\mathit{flipodds}(w))^c = \mathit{flipevens}(w)$. ■

Rubric: 4 points: Standard induction rubric (scaled)

(b) Prove for all strings $w, x \in \{0, 1\}^*$, we have

$$flipodds(w \bullet x) = \begin{cases} flipodds(w) \bullet flipodds(x) & \text{if } |w| \text{ is even} \\ flipodds(w) \bullet flipevens(x) & \text{if } |w| \text{ is odd.} \end{cases}$$

Solution: Let w be an arbitrary string.
Assume for every string $y \in \{0, 1\}^*$ such that $|y| < |w|$, we have

$$flipodds(y \bullet x) = \begin{cases} flipodds(y) \bullet flipodds(x) & \text{if } |y| \text{ is even} \\ flipodds(y) \bullet flipevens(x) & \text{if } |y| \text{ is odd.} \end{cases}$$

There are two cases to consider based on $|w|$, and they both have two subcases.

- Suppose $|w|$ is even.
  - If $w = \varepsilon$, then

$$\begin{aligned} flipodds(w \bullet x) &= flipodds(\varepsilon \bullet x) && \text{because } w = \varepsilon \\ &= flipodds(x) && \text{by definition of } (\bullet) \\ &= \varepsilon \bullet flipodds(x) && \text{by definition of } (\bullet) \\ &= flipodds(\varepsilon) \bullet flipodds(x) && \text{by definition of } flipodds \\ &= flipodds(w) \bullet flipodds(x) && \text{because } w = \varepsilon \end{aligned}$$

  - If $w = aby$, then $|y|$ is even and

$$\begin{aligned} flipodds(w \bullet x) &= flipodds((aby) \bullet x) && \text{because } w = aby \\ &= flipodds(a \cdot (b \cdot (y \bullet x))) && \text{by definition of } (\bullet) \text{ x2} \\ &= \bar{a} \cdot (b \cdot flipodds(y \bullet x)) && \text{by definition of } flipodds \text{ x2} \\ &= \bar{a} \cdot (b \cdot (flipodds(y) \bullet flipodds(x))) && \text{by the inductive hypothesis} \\ &= (\bar{a} \cdot (b \cdot flipodds(y))) \bullet flipodds(x) && \text{by definition of } (\bullet) \text{ x2} \\ &= flipodds(aby) \bullet flipodds(x) && \text{by definition of } flipodds \text{ x2} \\ &= flipodds(w) \bullet flipodds(x) && \text{because } w = aby \end{aligned}$$

  In both cases, $flipodds(w \bullet x) = flipodds(w) \bullet flipodds(x)$.
- Suppose $|w|$ is odd.
  - If $w = a$, then

$$\begin{aligned} flipodds(w \bullet x) &= flipodds(a \bullet x) && \text{because } w = a \\ &= flipodds(a \cdot x) && \text{by definition of } (\bullet) \text{ x2} \\ &= \bar{a} \cdot flipevens(x) && \text{by definition of } flipodds \\ &= \bar{a} \bullet flipevens(x) && \text{by definition of } (\bullet) \text{ x2} \\ &= flipodds(a) \bullet flipevens(x) && \text{by definition of } flipodds \text{ x2} \\ &= flipodds(w) \bullet flipevens(x) && \text{because } w = a \end{aligned}$$

- If $w = aby$, then $|y|$ is odd and

$$\begin{aligned}
\mathit{flipodds}(w \bullet x) &= \mathit{flipodds}((aby) \bullet x) && \text{because } w = aby\\
&= \mathit{flipodds}(a \cdot (b \cdot (y \bullet x))) && \text{by definition of } (\bullet) \text{ x2}\\
&= \bar{a} \cdot (b \cdot \mathit{flipodds}(y \bullet x)) && \text{by definition of } \mathit{flipodds} \text{ x2}\\
&= \bar{a} \cdot (b \cdot (\mathit{flipodds}(y) \bullet \mathit{flipevens}(x))) && \text{by the inductive hypothesis}\\
&= (\bar{a} \cdot (b \cdot \mathit{flipodds}(y))) \bullet \mathit{flipevens}(x) && \text{by definition of } (\bullet) \text{ x2}\\
&= \mathit{flipodds}(aby) \bullet \mathit{flipevens}(x) && \text{by definition of } \mathit{flipodds} \text{ x2}\\
&= \mathit{flipodds}(w) \bullet \mathit{flipevens}(x) && \text{because } w = aby
\end{aligned}$$

In both cases, $\mathit{flipodds}(w \bullet x) = \mathit{flipodds}(w) \bullet \mathit{flipevens}(x)$.

■

Rubric: 6 points: Standard induction rubric (scaled)

2. Consider the following 2 sets of strings $A, B \subseteq \{0, 1\}^*$ defined (mutually) recursively as follows:

   - The empty string ε is in A.
   - For any string x in A, the string $0x$ is in B, and the string $1x$ is in A.
   - For any string x in B, the string $0x$ is in A, and the string $1x$ is in B.
   - These are the only strings in A and B.

   (a) Prove that for every string $w \in \{0, 1\}^*$

   - if $\#(0, w)$ is even, then $w \in A$, and
   - if $\#(0, w)$ is odd, then $w \in B$.

Solution: Let w be an arbitrary string in $\{0, 1\}^*$.
Assume, for any string $x \in \{0, 1\}^*$ with $|x| < |w|$ that

- if $\#(0, x)$ is even, then $x \in A$, and
- if $\#(0, x)$ is odd, then $x \in B$.

Suppose $\#(0, w)$ is even. There are three cases to consider.

- Suppose $w = \varepsilon$. Then, $w \in A$ be definition.
- Suppose $w = 0x$. Then, $\#(0, x)$ is odd. By the inductive hypothesis, $x \in B$. Therefore, $w = 0x$ is in A by definition.
- Suppose $w = 1x$. Then, $\#(0, x)$ is even. By the inductive hypothesis, $x \in A$. Therefore, $w = 1x$ is in A by definition.

In all cases, $w \in A$.
Now, suppose $\#(0, w)$ is odd. There are two cases to consider.

- Suppose $w = 0x$. Then, $\#(0, x)$ is even. By the inductive hypothesis, $x \in A$. Therefore, $w = 0x$ is in B by definition.
- Suppose $w = 1x$. Then, $\#(0, x)$ is odd. By the inductive hypothesis, $x \in B$. Therefore, $w = 1x)$ is in B by definition.

In both cases, $w \in B$. ■

Rubric: 5 points: standard induction rubric (scaled)

(b) Prove that for every string $w \in \{\mathtt{0}, \mathtt{1}\}^*$

- if $w \in A$, then $\#(\mathtt{0}, w)$ is even, and
- if $w \in B$, then $\#(\mathtt{0}, w)$ is odd.

Solution: Let w be an arbitrary string in $\{\mathtt{0}, \mathtt{1}\}^*$.

Assume, for any string $x \in \{\mathtt{0}, \mathtt{1}\}^*$ with $|x| < |w|$ that

- if $x \in A$, then $\#(\mathtt{0}, x)$ is even, and
- if $x \in B$, then $\#(\mathtt{0}, x)$ is odd.

Suppose $w \in A$. There are three cases to consider.

- Suppose $w = \varepsilon$. Then, $\#(0, w) = 0$ by definition of $\#$, and that value is even.
- Suppose $w = \mathtt{0}x$. Then, $x \in B$ by definition. By the inductive hypothesis $\#(\mathtt{0}, x)$ is odd. By the definition of $\#$, we have $\#(\mathtt{0}, w) = \#(\mathtt{0}, \mathtt{0}x)$ is even.
- Suppose $w = \mathtt{1}x$. Then, $x \in A$ by definition. By the inductive hypothesis $\#(\mathtt{0}, x)$ is even. By the definition of $\#$, we have $\#(\mathtt{0}, w) = \#(\mathtt{0}, \mathtt{1}x)$ is even.

In all cases, $\#(\mathtt{0}, w)$ is even.

Now, suppose $w \in B$. There are two cases to consider.

- Suppose $w = \mathtt{0}x$. Then, $x \in A$ by definition. By the inductive hypothesis $\#(\mathtt{0}, x)$ is even. By the definition of $\#$, we have $\#(\mathtt{0}, w) = \#(\mathtt{0}, \mathtt{0}x)$ is odd.
- Suppose $w = \mathtt{1}x$. Then, $x \in B$ by definition. By the inductive hypothesis $\#(\mathtt{0}, x)$ is odd. By the definition of $\#$, we have $\#(\mathtt{0}, w) = \#(\mathtt{0}, \mathtt{1}x)$ is odd.

In all cases, $\#(\mathtt{0}, w)$ is odd. ■

Rubric: 5 points: standard induction rubric (scaled)

*3. *Practice only. Do not submit solutions.*

For each non-negative integer n, we recursively define two binary trees P_n and V_n, called the nth *Piṅgala tree* and the nth *Virahāṇka tree*, respectively.

- P_0 and V_0 are empty trees, with no nodes.
- P_1 and V_1 each consist of a single node.
- For any integer $n \geq 2$, the tree P_n consists of a root with two subtrees; the left subtree is a copy of P_{n-1}, and the right subtree is a copy of P_{n-2}.
- For any integer $n \geq 2$, the tree V_n is obtained from V_{n-1} by attaching a new right child to every leaf and attaching a new left child to every node that has only a right child.

(a) Prove that the tree P_n has exactly F_n leaves.

Solution: To make the presentation simpler, let me define some notation. Let ε denote the empty binary tree, and let (L, R) denote the non-empty binary tree with left subtree L and right subtree R. Let $\bullet = (\varepsilon, \varepsilon)$ denote the binary tree consisting of a single node. Then we can define Piṅgala trees more succinctly as follows:

$$P_n = \begin{cases} \varepsilon & \text{if } n = 0 \\ \bullet & \text{if } n = 1 \\ (P_{n-2}, P_{n-1}) & \text{otherwise} \end{cases}$$

Now we're ready for the proof:

Proof: Let n be an arbitrary non-negative integer.
Assume $\#\text{leaves}(P_m) = F_m$ for every non-negative integer $m < n$.
There are three cases to consider.

- Suppose $n = 0$. $P_0 = \varepsilon$ has no nodes, so $\#\text{leaves}(P_0) = 0 = F_0$.
- Suppose $n = 1$. The single node in $P_1 = \bullet$ is a leaf, so $\#\text{leaves}(P_1) = 1 = F_1$.
- Finally, suppose $n \geq 2$. The root of P_n is not a leaf, so

$$\begin{aligned} \#\text{leaves}(P_n) &= \#\text{leaves}(\,(P_{n-2}, P_{n-1})\,) && \text{by definition of } P_n \\ &= \#\text{leaves}(P_{n-2}) + \#\text{leaves}(P_{n-1}) && \text{root of } P_n \text{ is not a leaf} \\ &= F_{n-2} + F_{n-1} && \text{by ind. hyp.} \\ &= F_n && \text{by definition of } F_n \end{aligned}$$

In all cases, we conclude that $\#\text{leaves}(P_n) = F_n$. ■

Rubric: Standard induction rubric. Weak induction *cannot* work here.

(b) Prove that the tree V_n has exactly F_n leaves.
[Hint: You need to prove a stronger result.]

Solution: To simplify the presentation, let $\#(d, T)$ denote the number of nodes in the tree T with exactly d children. In particular, $\#(0, T)$ is the number of leaves in T. We need to prove that $\#(0, V_n) = F_n$ for all $n \geq 0$.

This claim is trivial when $n = 0$. For all $n > 0$, I'll actually prove the stronger claim that $\#(0, V_n) = F_n$ and $\#(1, V_n) = F_{n-1}$.

Let n be an arbitrary positive integer. Assume for all positive integers $m < n$ that $\#(0, V_m) = F_m$ and $\#(1, V_m) = F_{m-1}$. There are two cases to consider, mirroring the definition of V_n.

- Suppose $n = 1$. Then by definition, V_1 has a single node, which is a leaf, so
$$\begin{aligned} \#(0, V_n) &= \#(0, V_1) = 1 = F_1 = F_n \\ \#(1, V_n) &= \#(1, V_1) = 0 = F_0 = F_{n-1} \end{aligned}$$
- Suppose $n \geq 2$. Then each leaf of V_n is a child of either a leaf of V_{n-1} or a node with one child in V_{n-1}. Conversely, each node with zero or one children in V_{n-1} is the parent of exactly one leaf in V_n. So the inductive hypothesis implies
$$\#(0, V_n) = \#(0, V_{n-1}) + \#(1, V_{n-1}) = F_{n-1} + F_{n-2} = F_n.$$
Similarly, each node with one child in V_n is a leaf in V_{n-1}, so the inductive hypothesis implies $\#(1, V_n) = \#(0, V_{n-1}) = F_{n-1}$.

In both cases, we conclude that $\#(0, V_n) = F_n$ and $\#(1, V_n) = F_{n-1}$. ■

Rubric: Standard induction rubric. We do need to consider the case $n = 0$ outside the main induction argument; our stronger claim is false when $n = 0$, because $F_{-1} = 1$!

(c) Prove that the trees P_n and V_n are identical, for all $n \geq 0$.

Solution: Let *T.left* and *T.right* denote the left and right subtrees of any non-empty tree T. For any tree T, we recursively define

$$sprout(T) = \begin{cases} \bullet & \text{if } T = \varepsilon \\ (\varepsilon, \bullet) & \text{if } T = \bullet \\ (sprout(T.left), sprout(T.right)) & \text{otherwise} \end{cases}$$

Then we can rewrite the definition of V_n as $V_n = sprout(V_{n-1})$ for all $n \geq 1$.

Let n be an arbitrary non-negative integer. Assume for all non-negative integers $m < n$ that $P_n = V_n$. There are four cases to consider.

- $P_0 = \varepsilon$ and $V_0 = \varepsilon$ by definition.
- $P_1 = \bullet$ and $V_1 = \bullet$ by definition.
- $P_2 = (P_0, P_1) = (\varepsilon, \bullet)$ and $V_2 = sprout(V_1) = sprout(\bullet) = (\varepsilon, \bullet)$ by definition.
- Finally, if $n \geq 3$, we have

$$\begin{aligned}
V_n &= sprout(V_{n-1}) && \text{by definition} \\
&= sprout(P_{n-1}) && \text{by the inductive hypothesis} \\
&= sprout(\,(P_{n-3}, P_{n-2})\,) && \text{by definition of } P_n \\
&= sprout(\,(V_{n-3}, V_{n-2})\,) && \text{by the inductive hypothesis (twice)} \\
&= (\,sprout(V_{n-3}), sprout(V_{n-2})\,) && \text{by definition of } sprout\text{, since } n \geq 4 \\
&= (V_{n-2}, V_{n-1}) && \text{by definition of } V_n \\
&= (P_{n-2}, P_{n-1}) && \text{by the inductive hypothesis (twice)} \\
&= P_n && \text{by definition of } P_n
\end{aligned}$$

In all cases, we conclude that $V_n = P_n$. ■

Rubric: Standard induction rubric. Weak induction cannot work here. We need to consider the case $n = 2$ separately, because the main inductive proof refers to V_{n-3} and P_{n-3}.

Watch for switching between P_m and V_m without explicitly invoking the induction hypothesis. Yes, we really do have to invoke the induction hypothesis five times!