HW 11 Tue Dec 1

GPS 11 Mon Dec 7

No HW party during Fall Break.

Final Conflict form & CA applications out soon!

Final exam Fri Dec 11 8am - 11am.

In a graph $G=(V,E)$, a Hamiltonian cycle is a cycle that includes every vertex exactly once.

DirHamCycle: Given directed $G=(V,E)$, does G have a Hamiltonian cycle?

Thm: DirHamCycle is NP-complete.

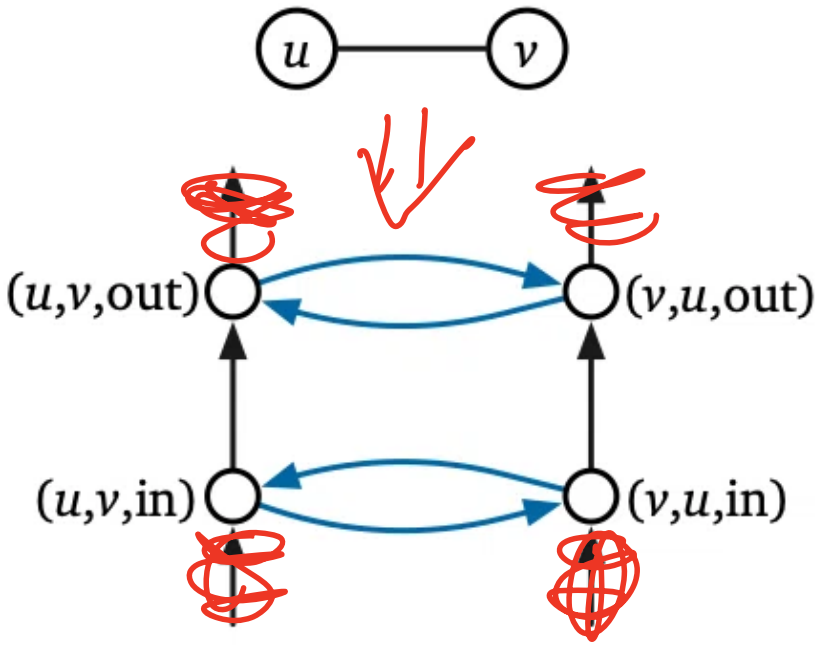
Proof: In NP: Certificate is the Ham. cycle.

Will show VertexCover $\leq_p$ DirHamCycle.

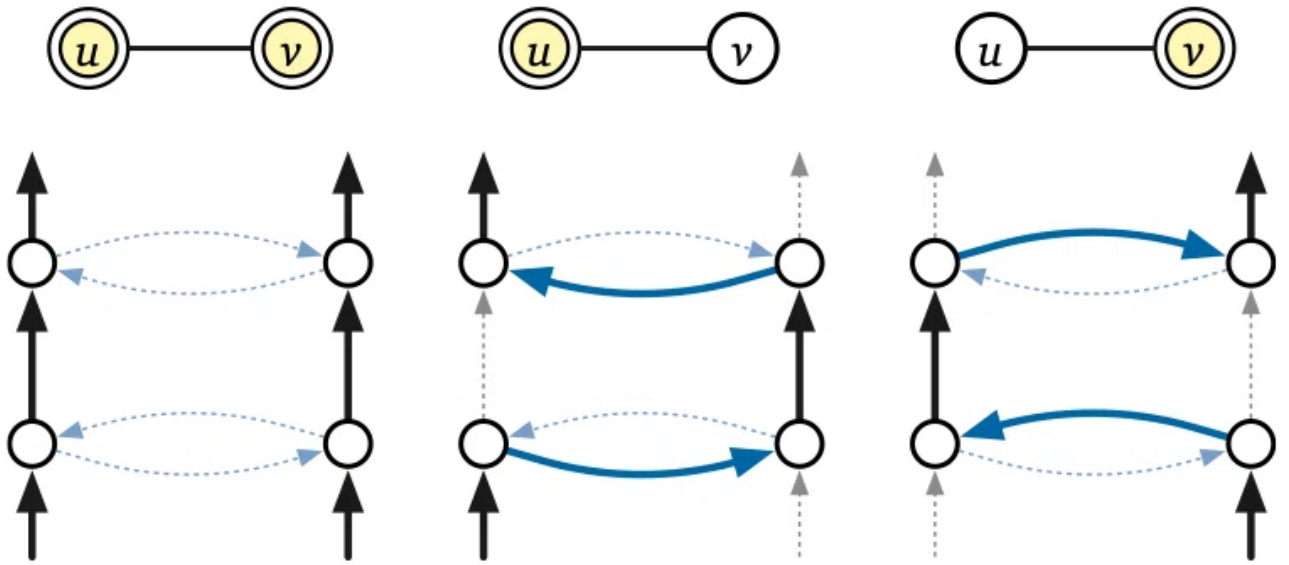
Given undirected graph $G=(V,E)$ & integer $k \geq 0$, build directed $H=(V',E')$.

For each edge $uv \in E$, add one edge gadget with four vertices (u, v, in)
(u, v, out)
(v, u, in)
(u, v, out)

+ six edges



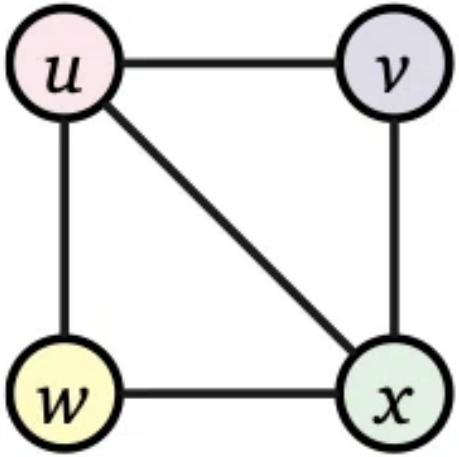
How a Ham cycle goes through tells us what covers uv.

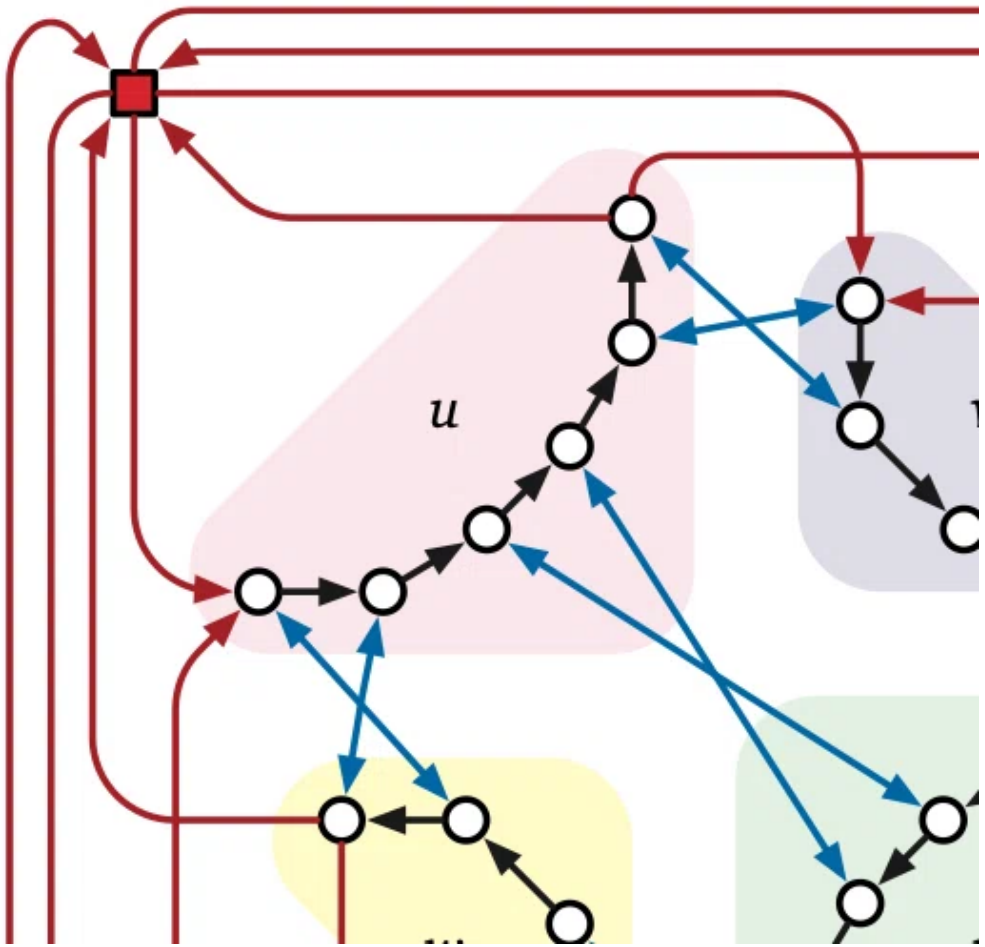


For each vertex $u \in V$, we add a "vertex chain" to H.

Say u has d neighbors $v_1, v_2, \ldots, v_d$.

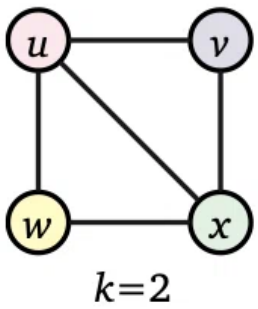
Add $d-1$ edges to H $(u, v_i, \text{out}) \Rightarrow (u, v_{i+1}, \text{in})$ for each $i \in \{1, \ldots, d-1\}$.

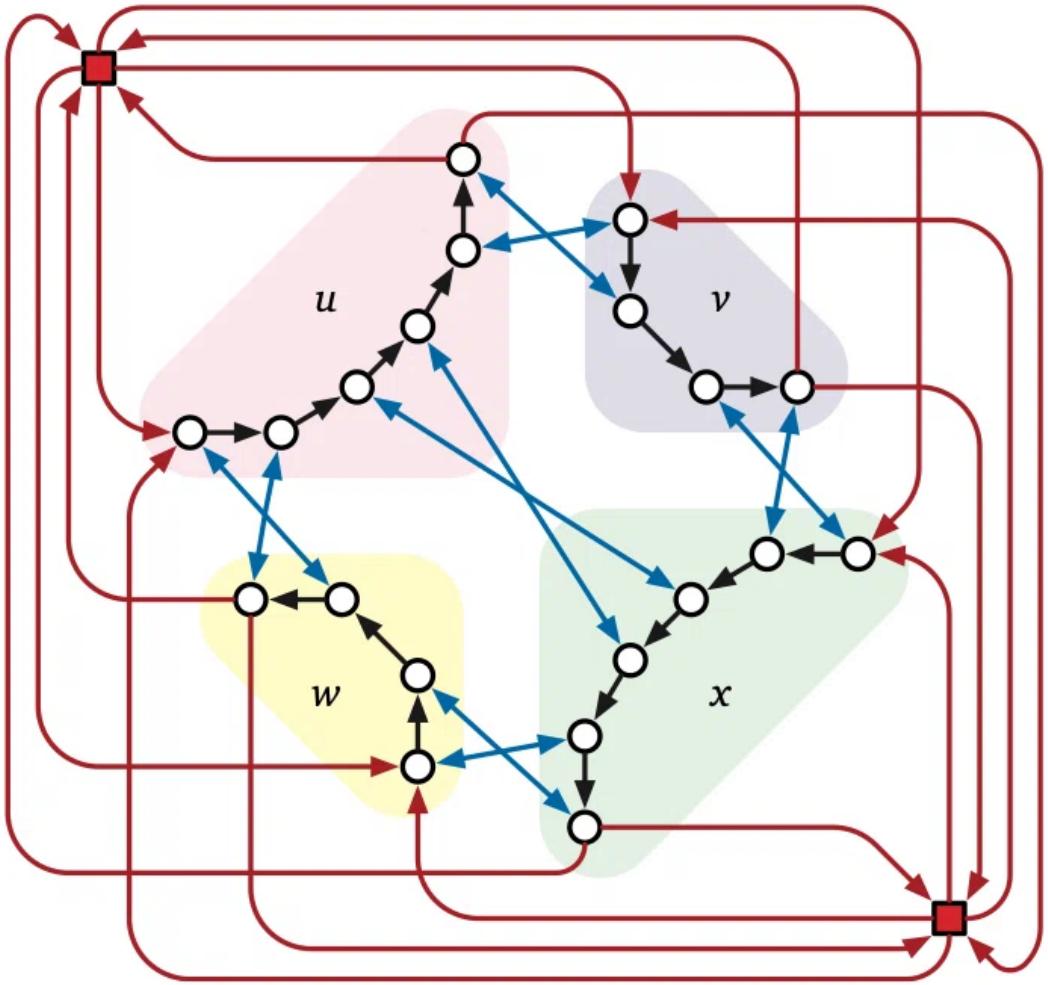




If a Ham cycle enters the chain for u at (u, v_i, in), it enters all u's edge gadgets at (u, v_i, in) & leaves the chain at (u, v_i, out).

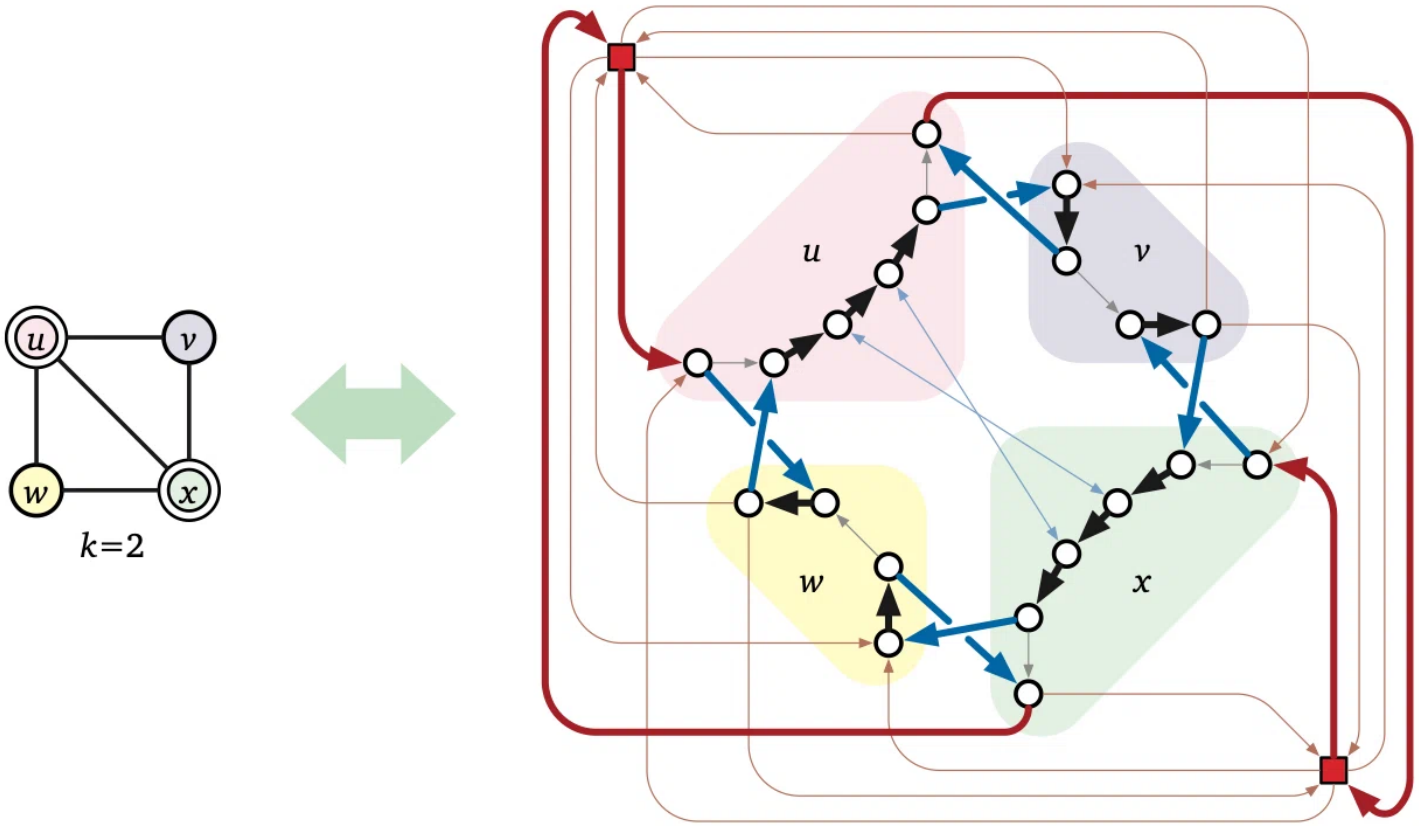
Add k cover vertices $x_1, \ldots, x_k$. Each has edges to (u, v_i, in) for all u & has edge from (u, v_i, out) for all u.





Takes polynomial $O(|V|+|E|)$ time.

We need to show G has a vertex cover of size k iff H has a Ham. cycle.



Show a cover $S \subseteq V$ leads to a cycle.

Show a cycle leads to a cover.

(see notes on website!)

Also, $DirHamCycle \leq_p UndirHamCycle$

Also, DirHamPath & UndirHamPath are NP-complete.

(Is there a path with every vertex exactly once?)

Subset Sum: Given a set X of positive integers & one non-negative integer T.

Need to decide is there a subset

$$S \subseteq X \quad \text{s.t.} \sum_{x \in S} x = T?$$

Thm: Subset Sum is NP-complete.

Proof: in NP: certificate is the subset S summing to T.

Will show Vertex Cover $\leq_p$ Subset Sum.

Given undirected $G = (V, E)$ & $k \geq 0$.

We'll build (X, T).

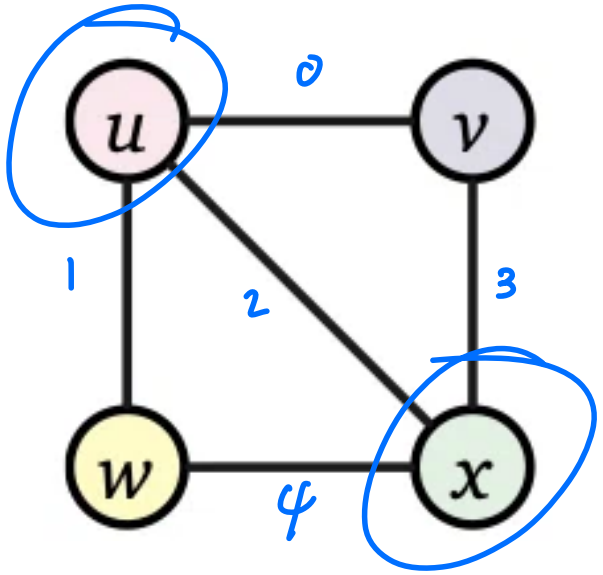
Arbitrarily number the edges of G from 0 to $|E|-1$.

For each edge i, insert integer $b_i := 4^i$ in X.

For each vertex v, let $\Delta(v)$ be its incident edges.

Insert $a_v := 4^{|E|} + \sum_{i \in \Delta(v)} 4^i$ to X.

Set $T := k \cdot 4^{|E|} + \sum^{|E|-1} 2 \cdot 4^i$



$i = 0$

$$a_u := 100111_4 \qquad b_{uv} := 000001_4$$
$$a_v := 101001_4 \qquad b_{uw} := 000010_4$$
$$a_w := 110010_4 \qquad b_{ux} := 000100_4$$
$$a_x := 111100_4 \qquad b_{vx} := 001000_4$$
$$b_{wx} := 010000_4$$

$k = 2$

$$T := 2 \cdot 100000_4 + 022222_4$$

Takes polynomial $O(|V||E|)$ time.

Need to show G has a vertex cover of size k iff a subset of X sums to T.

$\Rightarrow$ Suppose G has a cover C of size

k. Let

$S = \{a_v \mid v \in C\} \cup \{b_i \mid$ edge i has one endpoint in $C\}$

For all $i \le |E|$, two members of S use crumb i. $|C| = k$ so we also get the $k \cdot 4^{|E|}$.

$\Leftarrow$ Let S be any subset summing to T. For all $i \le |E|$, only 3 members of X use crumb i, so we took one or two incident vertex number & maybe b_i. To get the $k \cdot 4^{|E|}$ we took k vertex

numbers. Those vertices are a vertex cover.

Can solve Subset Sum in $O(nT)$ time where $n = |X|$.

exponential in input size

Subset Sum has a pseudo-polynomial time algorithm.

Subset Sum is weakly NP-hard.

Had to use exponentially large numbers.

To prove B is hard, choose a problem to reduce from

- Binary choices:
  - 3SAT
- Large subsets:
  - IndependentSet
  - Clique
- Small subsets:
  - VertexCover
  - SetCover (from VertexCover)
    - Given a set X and several subsets of X, pick as few subsets as possible to cover all members of X.
- Partition into **3** or more subsets:
  - 3Color
- Order things or make pretty subgraphs:
  - Directed/Undirected Hamiltonian Cycle/Path
  - Long Cycle/Path (from Hamiltonian Cycle/Path)
  - Traveling Salesman Problem/World Tour (from Undirected Hamiltonian Cycle)
- Select “balanced” subsets
  - SubsetSum
  - Partition (from SubsetSum)
    - Can a given subset of integers be partitioned into two subsets with equal sums?
  - 3DMatching/Exact3DMatching/X3M (from 3SAT)
    - Given a three sets A, B, and C of n elements each along with triples from $A \times B \times C$, can you pick exactly n triples to cover every element?
  - 3Partition (from 3DMatching)
    - Given $3n$ integers, can they be partitioned into n subsets of size 3

Need to take any instance of start problem & transform to some instance of new problem.

Start with as simple a problem as you can. Make a special new instance.