HW 10: today, Tue Nov 17

HW 11: Tue, Dec 1

GPS 11: Mon Dec 7

No HW Parties over Break.

SAT: Given a CNF Boolean formula, can we set variables to make formula True?

SAT is NP-complete (Cook-Levin)

To prove a problem B is NP-hard, reduce a known NP-hard problem A to problem B.

3SAT (uses 3 literals

is NP-complete in each clause)

Ex: Independent Set (IS)

Given undirected $G = (V, E)$ & k. Want to know, does G contain an ind. set ($S \subseteq V$ where no two vertices in S share an edge of G) of size k?

Thm: Independent Set is NP-complete.

$IS \subseteq NP$: certificate is S itself

Will reduce 3SAT to IS.

(i.e., show $3SAT \leq_p IS$)

Let ϕ be a 3CNF Boolean formula.

ex: $(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$

Let n: # variables in ϕ

k: # clauses in ϕ

Build undirected $G = (V, E)$ & k' (an instance of IS).

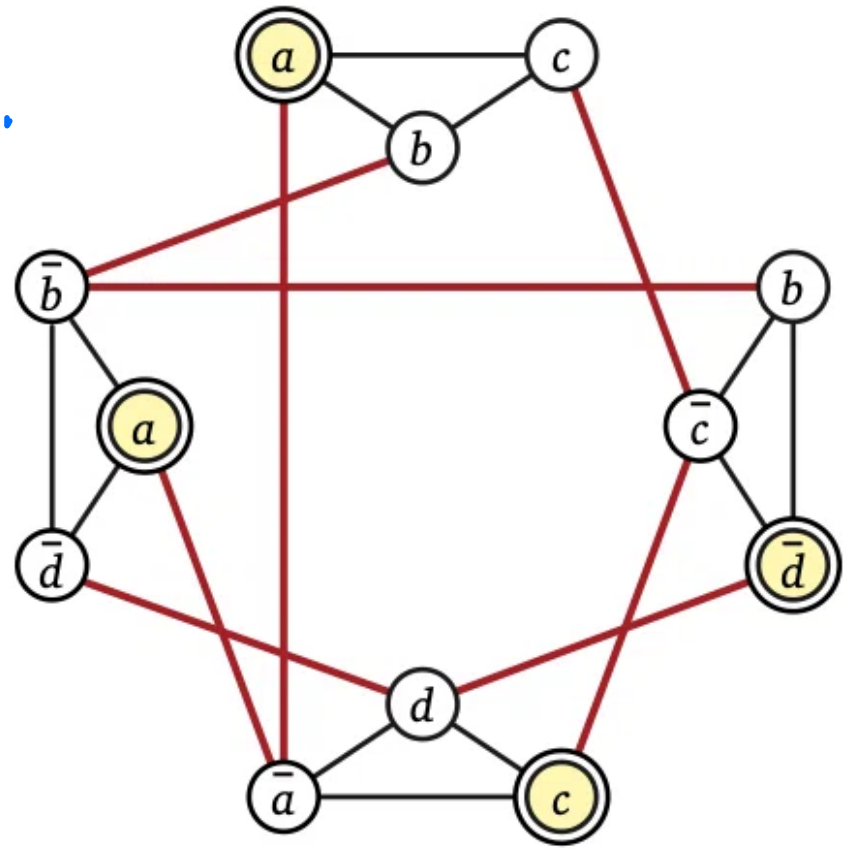
G has $3k$ vertices, the literals of ϕ.

G has an edge uv iff either

- u & v are from the same clause —
- u & v are opposite literals of the same variable —

$$(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$$

set $k' := k$.



sat. using

$a = b = c =$ True

$d =$ False

Takes polynomial $O(k)$ time.

Need to prove ϕ is satisfiable iff G has an ind. set of size $k' = k$.

$\Rightarrow$ Suppose there is a satisfying assignment for ϕ.

Let $S \subseteq V$ be one true literal per clause.

$|S| = k = k'$.

No — edges in S: one vertex per clause

No — edges in S: only one of x or $\bar{x}$ is a true literal

So S is independent.

$\Leftarrow$ Suppose there is an ind. set S of size k.

S has one vertex per triangle (one literal per clause)

$\Rightarrow$ S has $\geq$ one vertex per triangle, because k triangles

S cannot have both a literal x & its negation. (—)

For each literal x or $\bar{x}$ in S, set x to satisfy that literal.

We satisf., ϕ.

□

Cor: Clique & VertexCover are NP-complete
(reductions shown last week)

A proper k-coloring of undirected $G = (V, E)$ is an assignment $C : V \to \{1, \ldots, k\}$ s.t. for all $uv \in E$, $C(u) \neq C(v)$.

3Color: Given undirected $G = (V, E)$, does it have a proper 3-coloring?

Thm: 3Color is NP-complete

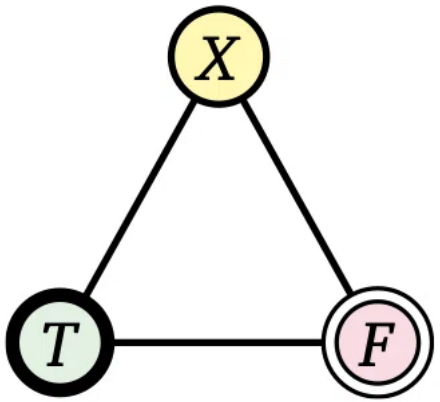
Proof: 3Color $\in$ NP: Certificate is the proper 3-coloring.

3Color $\in$ NP-hard: Reduce <u>from</u> 3SAT.
(3SAT $\leq_p$ 3Color)

Given 3CNF Boolean formula ϕ.

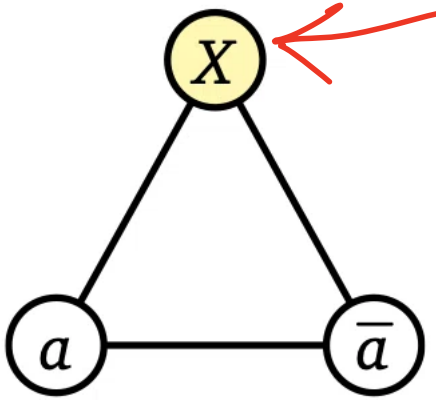
Build $G = (V, E)$...

G has <u>a</u> truth gadget:
a triangle with vertices T, F, & X.



True
False
Other

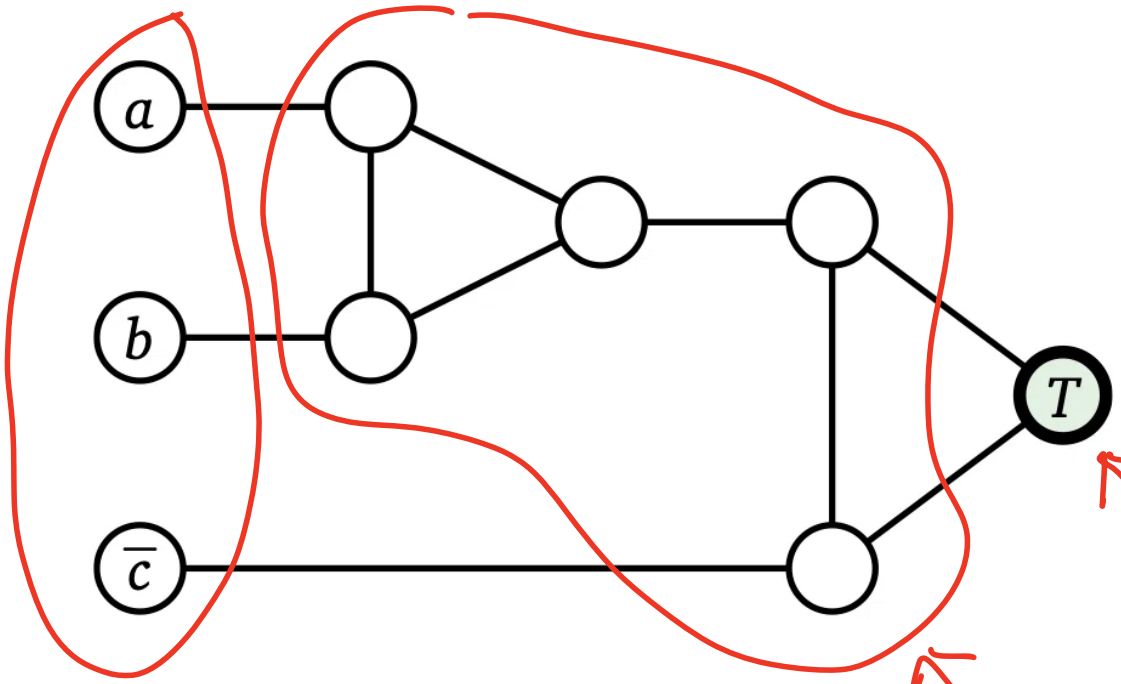
For each variable a of ϕ, create a variable gadget,



same X as before

If X gets color Other, a & $\bar{a}$ get both True & False.

For each clause $(x \vee y \vee z)$, add a clause gadget



$\leftarrow (a \vee b \vee \bar{c})$

same as truth gadget

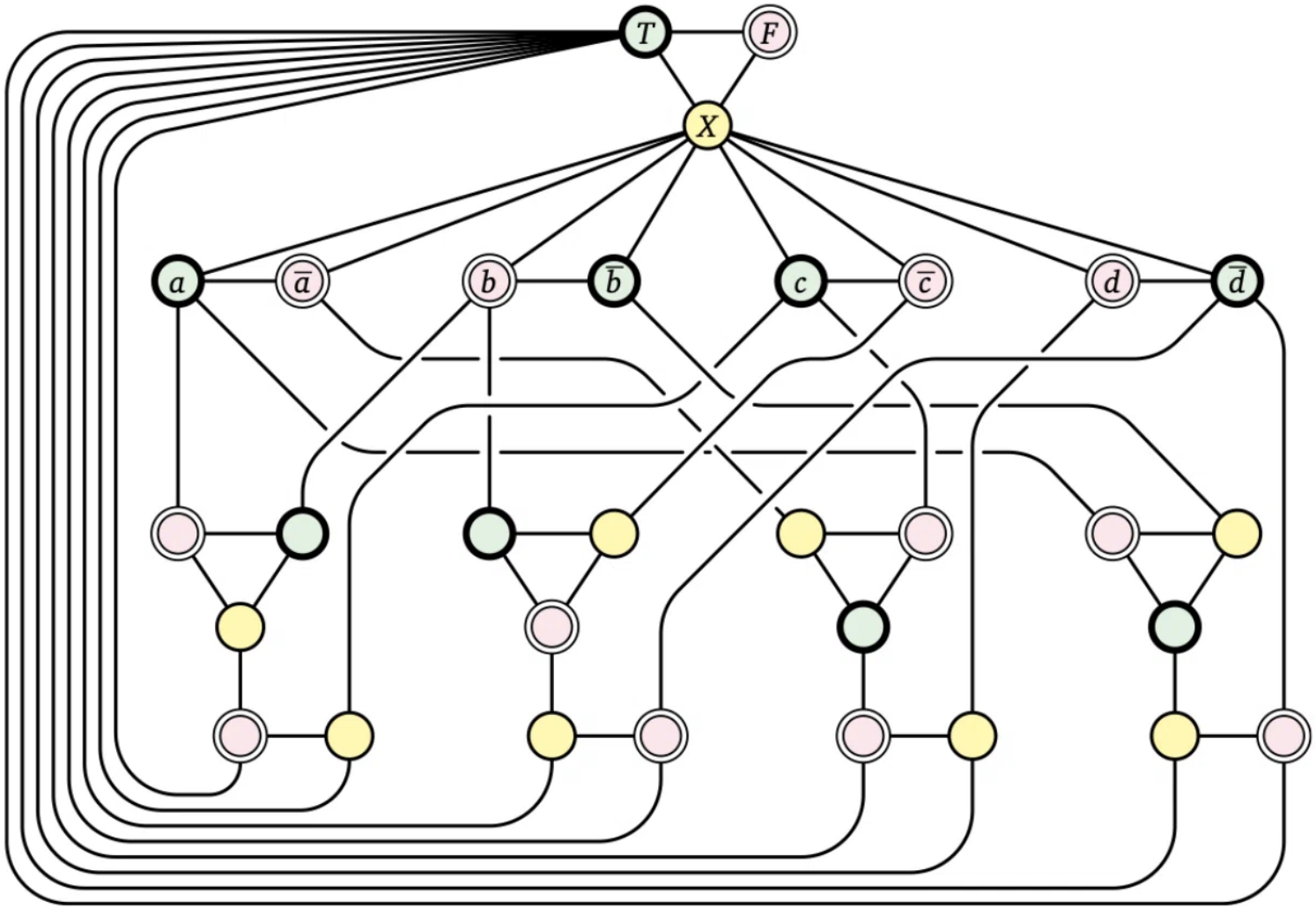
same as variable gadgets

unique to this clause gadget

At least one of the literal vertices

must get True to properly 3-color the gadget.

$$(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d}).$$



Takes polynomial $O(n)$ time.

Need to show ϕ is satisfiable iff G has a proper 3-coloring.

$\Rightarrow$

$\Leftarrow$