

1. A **basic arithmetic expression** is a string over the alphabet $\{1, +, \times, (,)\}$ that represents a properly bracketed and unambiguous arithmetic expression. Formally, a basic arithmetic expression is any string generated by the following context-free grammar (with starting non-terminal A):

$$\begin{array}{ll} A \rightarrow S \mid P & \text{[Arithmetic expression]} \\ S \rightarrow 1 \mid (P) \mid S+S & \text{[Sum]} \\ P \rightarrow 1 \mid (S) \mid P \times P & \text{[Product]} \end{array}$$

Every basic arithmetic expression has a positive integer *value*, which can be computed by interpreting $+$ as addition and $\times$ as multiplication, exactly as you would expect. Almost every integer is the value of more than one basic arithmetic expression. For example, all of the following basic arithmetic expressions have value 14:

$$\begin{aligned} &1+1+1+1+1+1+1+1+1+1+1+1+1+1 \\ &((1+1) \times (1+1+1+1)) + ((1+1) \times (1+1)) \\ &(1+1) \times (1+1+1+1+1+1) \\ &(1+1) \times (((1+1+1) \times (1+1)) + 1) \end{aligned}$$

Describe and analyze an algorithm to compute, given an integer n as input, the minimum number of 1 s in a basic arithmetic expression whose value is n . The number of parentheses doesn't matter, just the number of 1 s. For example, when $n = 14$, your algorithm should return the integer 8, which is the number of 1 s in the final expression above. The running time of your algorithm should be polynomial in n .

Harder problem to think about later:

2. Suppose you are given a sequence of integers separated by $+$ and $-$ signs; for example:

$$1 + 3 - 2 - 5 + 1 - 6 + 7$$

Both addition and subtraction are left-associative, so this expression evaluates to -1 . You can change the expression's value by inserting parentheses in different places. For example:

$$\begin{aligned} 1 + 3 - 2 - 5 + 1 - (6 + 7) &= -15 \\ (1 + 3 - (2 - 5)) + (1 - 6) + 7 &= 9 \\ (1 + (3 - 2)) - (5 + 1) - (6 + 7) &= -17 \end{aligned}$$

Describe and analyze an algorithm to compute the *largest* possible value the expression can take by adding parentheses. Parentheses must be used only to group additions and subtractions; in particular, do not use them to create implicit multiplication as in $1 + 3(-2)(-5) + 1 - 6 + 7 = 33$.