1. Suppose that you have just finished computing the array $dist[1..|V|, 1..|V|]$ of shortest-path distances between ***all*** pairs of vertices in an edge-weighted directed graph $G = (V, E)$. Unfortunately, you discover that you incorrectly entered the weight of a single edge $u \to v$, so all that precious CPU time was wasted. Or was it? Maybe your distances are correct after all!

   In each of the following problems, let $w(u \to v)$ denote the weight that you used in your distance computation, and let $w'(u \to v)$ denote the correct weight of $u \to v$. You may assume there are no negative weight cycles in G with either weight $w(u \to v)$ or $w'(u \to v)$.

   (a) Suppose $w(u \to v) > w'(u \to v)$; that is, the weight you used for $u \to v$ was *larger* than its true weight. Describe an algorithm that repairs the distance array in $O(|V|^2)$ ***time*** under this assumption. *[Hint: For every pair of vertices x and y, either $u \to v$ is on the shortest path from x to y or it isn't.]*

   (b) Maybe even that was too much work. Describe an algorithm that determines whether your original distance array is actually correct in $O(1)$ ***time***, again assuming that $w(u \to v) > w'(u \to v)$. *[Hint: Either $u \to v$ is the shortest path from u to v or it isn't.]*

   (c) **To think about later:** Describe an algorithm that determines in $O(|E||V|)$ ***time*** whether your distance array is actually correct, even if $w(u \to v) < w'(u \to v)$.

   (d) **To think about later:** Argue that when $w(u \to v) < w'(u \to v)$, repairing the distance array *requires* recomputing shortest paths from scratch, at least in the worst case.

2. You—yes, *you*—can cause a major economic collapse with the power of algorithms![1] The *arbitrage* business is a money-making scheme that takes advantage of differences in currency exchange. In particular, suppose that 1 US dollar buys 120 Japanese yen; 1 yen buys 0.01 euros; and 1 euro buys 1.2 US dollars. Then, a trader starting with $1 can convert their money from dollars to yen, then from yen to euros, and finally from euros back to dollars, ending with $1.44!. The cycles of currencies $\$ \to ¥ \to € \to \$$ is called an ***arbitrage cycle***. Of course, finding an exploiting arbitrage cycles before the prices are corrected requires extremely fast algorithms.

   Suppose n different currencies are traded in your currency market. You are given a two-dimensional array $R[1..n, 1..n]$ containing exchange rates between every pair of currencies; for each i and j, one unit of currency i can be traded for $R[i, j]$ units of currency j. (Do *not* assume that $R[i, j] \cdot R[j, i] = 1$.)

   (a) Describe an algorithm that returns an array $MaxCurrency[1..n]$ where $MaxCurrency[1..n]$ is the maximum amount of currency i that you can obtain by trading starting with one unit of currency 1 assuming there are no arbitrage cycles.

   (b) Describe an algorithm to determine whether the given matrix of currency exchange rates creates an arbitrage cycle.

   (c) **To think about later:** Modify your algorithm from part (b) to actually return an arbitrage cycle if such a cycle exists.

[1] No, you can't.