Here are the formal recursive definitions of string length, concatenation, and reversal:

$$|w| := \begin{cases} 0 & \text{if } w = \varepsilon \\ 1 + |x| & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

$$w \cdot z := \begin{cases} z & \text{if } w = \varepsilon \\ a \cdot (x \cdot z) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

$$w^R := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ x^R \cdot a & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

Lemma 1: $w \cdot \varepsilon = w$ for all strings w.

Lemma 2: $|w \cdot z| = |w| + |z|$ for all strings w and z.

Lemma 3: $(w \cdot y) \cdot z = w \cdot (y \cdot z)$ for all strings w, y, and z.

1. Prove that $|w^R| = |w|$ for every string w.

Solution (induction on w):

Let w be an arbitrary string.

Assume for every string x where |x| < |w| that $|x^R| = |x|$.

There are two cases to consider.

• If $w = \varepsilon$, then

$$|w^R| = |\varepsilon^R|$$
 because $w = \varepsilon$
 $= |\varepsilon|$ by definition of ε^R
 $= |w|$ because $w = \varepsilon$

• Otherwise, w = ax for some symbol a and some string x, and therefore

$$|w^R| = |(ax)^R|$$
 because $w = ax$
 $= |x^R \cdot a|$ by definition of R
 $= |x^R| + |a|$ by Lemma 2
 $= |x^R| + 1 + |\varepsilon|$ by definition of $|\cdot|$
 $= |x^R| + 1$ by definition of $|\cdot|$
 $= |x| + 1$ by the induction hypothesis
 $= |ax|$ by definition of $|\cdot|$
 $= |w|$ because $w = ax$

In both cases, we conclude that $|w^R| = |w|$.

2. Prove that $(w \cdot z)^R = z^R \cdot w^R$ for all strings w and z.

Solution (induction on w):

Let w and z be arbitrary strings.

Assume for every string x where |x| < |w| that $(x \cdot z)^R = x^R \cdot z^R$.

There are two cases to consider:

• If $w = \varepsilon$, then

$$(w \cdot z)^R = (\varepsilon \cdot z)^R$$
 because $w = \varepsilon$
 $= z^R$ by definition of \bullet
 $= z^R \cdot \varepsilon$ by Lemma 1
 $= z^R \cdot \varepsilon^R$ by definition of \bullet
 $= z^R \cdot w^R$ because $w = \varepsilon$

• Otherwise, w = ax for some symbol a and some string x.

$$(w \cdot z)^{R} = (ax \cdot z)^{R}$$
 because $w = ax$

$$= (a \cdot (x \cdot z))^{R}$$
 by definition of •
$$= (x \cdot z)^{R} \cdot a$$
 by definition of R

$$= (z^{R} \cdot x^{R}) \cdot a$$
 by the induction hypothesis (*)
$$= z^{R} \cdot (x^{R} \cdot a)$$
 by Lemma 3
$$= z^{R} \cdot (ax)^{R}$$
 by definition of R

$$= z^{R} \cdot w^{R}$$
 because $w = ax$

In both cases, we conclude that $(w \cdot z)^R = z^R \cdot w^R$.

How did I know that the induction hypothesis needs to change the first string w, but not the second string z? I actually wrote down the inductive **argument** first, and then **noticed** that I needed to argue inductively about $x \cdot z$ at line (\star) . Same string z, but w changed to x.

Alternatively, we could notice that the recursive definition of $w \cdot z$ recurses on w but leaves z unchanged. Inductive proofs always mirror the recursive definitions of the objects in question.

Alternatively, in light of Lemma 2, we could have used induction on the **sum** of the string lengths. Then the inductive hypothesis would read "Assume for all strings x and y such that |x| + |y| < |w| + |z| that $(x \cdot y)^R = x^R \cdot y^R$."

3. Prove that $(w^R)^R = w$ for every string w.

Solution (induction on w):

Let *w* be an arbitrary string.

Assume for every string x where |x| < |w| that $(x^R)^R = x$.

There are two cases to consider.

• If $w = \varepsilon$, then

$$(w^R)^R = (\varepsilon^R)^R$$
 because $w = \varepsilon$
 $= \varepsilon^R$ by definition of ε^R
 $= \varepsilon$ by definition of ε^R
 $= w$ because $w = \varepsilon$

• Otherwise, w = ax for some symbol a and some string x.

$$(w^R)^R = ((ax)^R)^R$$
 because $w = ax$
 $= (x^R \cdot a)^R$ by definition of R
 $= a^R \cdot (x^R)^R$ by definition of R
 $= (\varepsilon^R \cdot a) \cdot (x^R)^R$ by definition of R
 $= (\varepsilon \cdot a) \cdot (x^R)^R$ by definition of R
 $= a \cdot (x^R)^R$ by definition of R
 $= a \cdot (x^R)^R$ by definition of R
 $= a \cdot x$ by the induction hypothesis
 $= w$ because $w = ax$

In both cases, we conclude that $(w^R)^R = w$.

To think about later: Let #(a, w) denote the number of times symbol a appears in string w. For example, #(X, WTF374) = 0 and #(0,0000101010101001000) = 12.

4. Give a formal recursive definition of #(a, w).

Solution:

$$\#(a,w) = \begin{cases} 0 & \text{if } w = \varepsilon \\ 1 + \#(a,x) & \text{if } w = ax \text{ for some string } x \\ \#(a,x) & \text{if } w = bx \text{ for some symbol } b \neq a \text{ and some string } x \end{cases}$$

Solution (clever notation):

$$\#(a, w) = \begin{cases} 0 & \text{if } w = \varepsilon \\ \#(a, x) + [a = b] & \text{if } w = bx \text{ for some symbol } b \text{ and some string } x \end{cases}$$

The expression [a = b] in red is Iverson bracket notation. For any proposition P, the expression [P] is equal to 1 is P is true and 0 is P is false.

5. Prove that $\#(a, w \cdot z) = \#(a, w) + \#(a, z)$ for all symbols a and all strings w and z.

Solution (induction on w):

Let a be an arbitrary symbol, and let w and z be arbitrary strings. Assume for any string x such that |x| < |w| that $\#(a, x \cdot z) = \#(a, x) + \#(a, z)$. There are three cases to consider.

• If $w = \varepsilon$, then

$$\#(a, w \cdot z) = \#(a, \varepsilon \cdot z)$$
 because $w = \varepsilon$
 $= \#(a, z)$ by definition of \bullet
 $= \#(a, \varepsilon) + \#(a, z)$ by definition of $\#(a, w) + \#(a, z)$ because $w = \varepsilon$

• If w = ax for some string x, then

$$\#(a, w \cdot z) = \#(a, ax \cdot z)$$
 because $w = ax$
 $= \#(a, a \cdot (x \cdot z))$ by definition of \bullet
 $= 1 + \#(a, x \cdot z)$ by definition of $\#$
 $= 1 + \#(a, x) + \#(a, z)$ by the induction hypothesis
 $= \#(a, ax) + \#(a, z)$ by definition of $\#$
 $= \#(a, w) + \#(a, z)$ because $w = ax$

• If w = bx for some symbol $b \neq a$ and some string x, then

$$\#(a, w \cdot z) = \#(a, bx \cdot z)$$
 because $w = bx$
 $= \#(a, b \cdot (x \cdot z))$ by definition of \bullet
 $= \#(a, x \cdot z)$ by definition of $\#(a, x) + \#(a, z)$ by the induction hypothesis
 $= \#(a, bx) + \#(a, z)$ by definition of $\#(a, w) + \#(a, z)$ because $w = bx$

In every case, we conclude that $\#(a, w \cdot z) = \#(a, w) + \#(a, z)$.

6. Prove that $\#(a, w^R) = \#(a, w)$ for all symbols a and all strings w.

Solution (induction on w): Let a be an arbitrary symbol, and let w be an arbitrary string.

Assume for any string x such that |x| < |w| that $\#(a, x^R) = \#(a, x)$.

There are three cases to consider.

- If $w = \varepsilon$, then $w^R = \varepsilon = w$ by definition, so $\#(a, w^R) = \#(a, w)$.
- If w = ax for some string x, then

$$\#(a, w^R) = \#(a, (ax)^R)$$
 because $w = ax$
 $= \#(a, x^R \cdot a)$ by definition of R
 $= \#(a, x^R) + \#(a, a)$ by problem 5
 $= \#(a, x^R) + 1$ by definition of R
 $= \#(a, x^R) + 1$ by definition of R
 $= \#(a, x^R) + 1$ by definition of R
 $= \#(a, x) + 1$ by the induction hypothesis
 $= \#(a, ax)$ by definition of R
 $= \#(a, ax)$ by definition of R
 $= \#(a, w)$ because R

• If w = bx for some symbol $b \neq a$ and some string x, then

$$\#(a, w^R) = \#(a, (bx)^R)$$
 because $w = bx$
 $= \#(a, x^R \cdot b)$ by definition of R
 $= \#(a, x^R) + \#(a, b)$ by definition of $\#$
 $= \#(a, x^R)$ by definition of $\#$
 $= \#(a, bx)$ by definition of $\#$
 $= \#(a, bx)$ by definition of $\#$
 $= \#(a, w)$ because $w = ax$

In every case, we conclude that $\#(a, w^R) = \#(a, w)$.