A language (decision problem) is <u>acceptable</u> if there is machine guaranteed to halt with correct Yes answers (but it may hang on No)

A language is <u>decidable</u> if there is a machine that always halts & correctly returns Yes or No.

(Undecidable o.w.)

Canonical undecidable language.

$$\text{Halt} := \{ \langle M, w \rangle \mid M \text{ halts on } w \}$$

Accept, Reject, Diverge also!

not even acceptable!

NeverHalt $:= \{\langle M \rangle \mid \mathrm{Halt}(M) = \emptyset\}$

Thm: NeverHalt is undecidable.

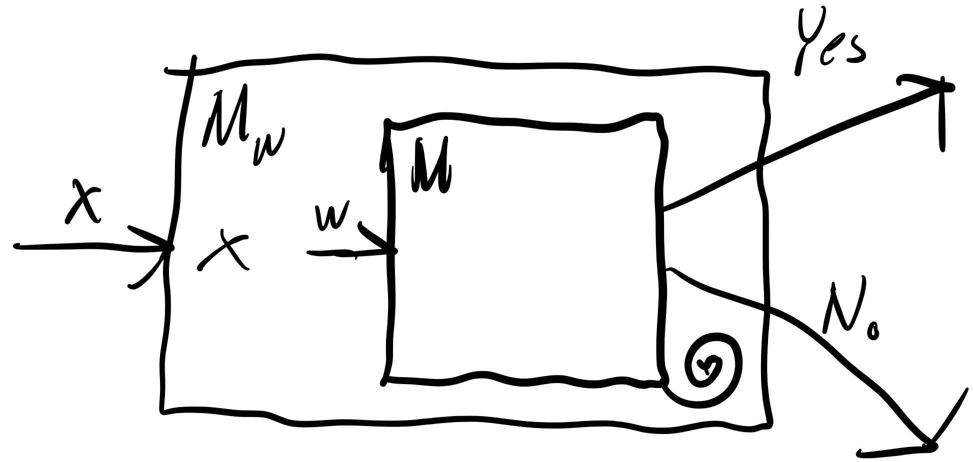
$\boxed{\text{To prove a language } L \text{ is undecidable, reduce a known undecidable language to } L.}$

Reduce from Halt.

Proof: Suppose there is a machine NH that decides NeverHalt.

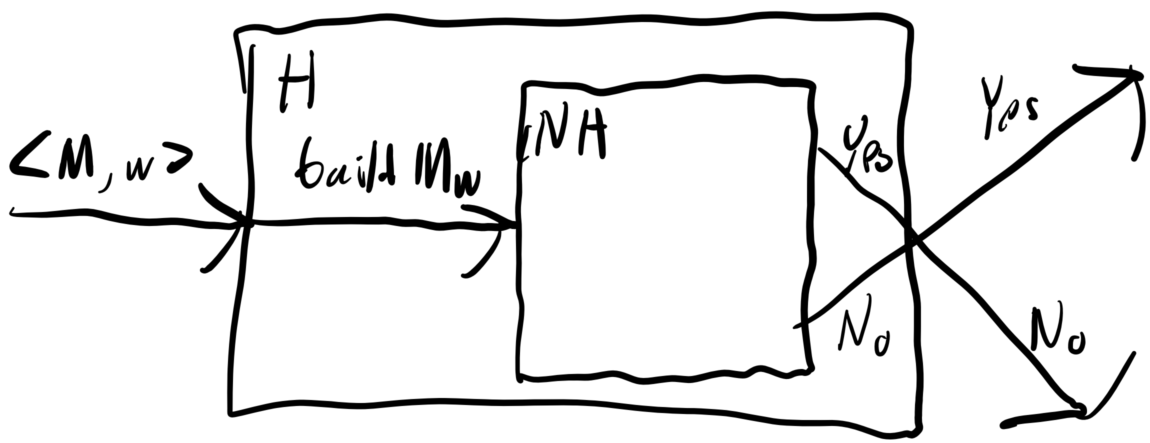
Given an instance <M, w> of Halt, let M_w be the following.

```
M_w(x):
    return M(w)
```



Build machine H:

```
H(<M, w>):
    return !NH(M_w)
```



Claim: H decides Halt.

Suppose M halts on w.

M_w halts on every input x.

So NH rejects $\langle M_w \rangle$

So H accepts $\langle M, w \rangle$.

Suppose M diverges on w.

M_w diverges on every input x.

So NH accepts $\langle M_w \rangle$

So H rejects $\langle M, w \rangle$.

But Halt is undecidable, contradicting existence of NH.

NeverAccept, NeverReject, NeverDiverge are also undecidable!

$\text{HaltSame} := \{ \langle M_1, M_2 \rangle \mid \underbrace{\text{Halt}(M_1)}_{\{w \mid M_1 \text{ halts on } w\}} = \text{Halt}(M_2) \}.$

Thm: HaltSame is undecidable.

Proof: (From NeverHalt to HaltSame)

Suppose there is a machine HS to decide HaltSame.

Let N be a machine that never halts.

(infinite loop?)

Build a machine NH for NeverHalt.

$$\boxed{\begin{array}{l} NH(\langle M \rangle): \\ \quad \text{return } HS(\langle M, N \rangle) \end{array}}$$

Claim: NH decides NeverHalt.

Suppose M never halts.

$Halt(M) = \emptyset = Halt(N)$

So HS accepts <M, N>

So NH accepts M.

Suppose M halts

$Halt(M) \neq \emptyset = Halt(N)$

so HS reject <M, N>

So NH rejects M.

But NH can't exist, so HS does not exist.

AcceptSame, RejectSame, DivergeSame also.

"Given $\langle M \rangle$, does M accept languages that ____ ?"

Rice's Theorem: Let $\mathcal{L}$ be any family of languages (set of sets of strings) such that

- there exists machine Y s.t. $\text{Accept}(Y) \in \mathcal{L}$.
- there exists machine N s.t. $\text{Accept}(N) \notin \mathcal{L}$.

Then $\text{AcceptIn}(\mathcal{L}) := \{\langle M \rangle \mid \text{Accept}(M) \in \mathcal{L}\}$ is undecidable.

Proof: Assume $\emptyset \notin \mathcal{L}$, and fix a machine Y s.t. $\text{Accept}(Y) \in \mathcal{L}$. (symmetry)

Suppose there is a machine $A_{\mathcal{L}}$ that decides $\text{AcceptIn}(\mathcal{L})$.

Given an instance $\langle M, w \rangle$ of Halt, let M_w^+

```
M_w^+(x):
  _ ← M(w)
  return Y(x)
```

Build H:

```
H(<M, w>):
  return A_L(<M_w^+>)
```

Suppose M halts on w:

For all x, $\mathrm{Accept}(M_w^+) = \mathrm{Accept}(Y)$

So $\mathrm{Accept}(M_w^+) \in \mathcal{L}$

So H accepts $\langle M, w \rangle$

Suppose M diverges on w:

M_w^+ always diverges (never finishes call to M)

$\mathrm{Accept}(M_w^+) = \emptyset \notin \mathcal{L}$

So $A_{\mathcal{L}}(\langle M_w^+ \rangle)$ rejects.

H rejects.

So H decides Halt. $\perp$ So no machine $A_{\mathcal{L}}$.

Thm: The following problems given a machine description $\langle M \rangle$ are undecidable:

Does M accept ε?

- $\mathcal{L}$: languages containing ε
- Y: accepts Σ^*
- N: rejects everything

Apply Rice's theorem.

Does M accept all palindrome of length 2^{prime}?

- $\mathcal{L}$: languages containing all palindromes of length 2^{prime}

- Y: accepts Σ^*
- N: reject everything

Does M accept a non-regular language?

- R: non-regular languages
- Y: accepts $\{0^n 1^n \mid n \geq 0\}$
- N: accepts Σ^* - or - accepts $\{“0”\}$