**Write your answers in the separate answer booklet.**
You have 180 minutes (after you get the answer booklet) to answer seven questions.

1. For each statement below, there are two boxes in the answer booklet labeled "Yes" and "No". Check "Yes" if the statement is ***always*** true and "No" otherwise, and give a ***brief*** (at most one short sentence) explanation of your answer. **Assume *P* ≠ *NP*.** If there is any other ambiguity or uncertainty about an answer, check "No". For example:

   - $x + y = 5$
     [ ] Yes [X] No — Suppose $x = 3$ and $y = 4$.
   - 3SAT can be solved in polynomial time.
     [ ] Yes [X] No — 3SAT is NP-hard.
   - If P = NP then Jeff is the Queen of England.
     [X] Yes [ ] No — The hypothesis is false, so the implication is true.

   Read each statement *very* carefully; some of these are deliberately subtle!

   (a) Which of the following statements are true for ***at least one*** language $L \subseteq \{0, 1\}^*$?
   - $L^* = (L^*)^*$
   - $L$ is decidable, but $L^*$ is undecidable.
   - $L$ is neither regular nor NP-hard.
   - $L$ is in P, and $L$ has an infinite fooling set.
   - The language $\{\langle M \rangle \mid M \text{ accepts } L\}$ is undecidable.

   (b) Suppose there is a ***polynomial-time*** reduction from some language $A$ over the alphabet $\{0, 1\}$ to some other language $B$ over the alphabet $\{0, 1\}$. Which of the following statements are ***always*** true, assuming P ≠ NP?
   - $A$ is a subset of $B$.
   - If $B \in P$, then $A \in P$.
   - If $B$ is NP-hard, then $A$ is NP-hard.
   - If $B$ is regular, then $A$ is regular.
   - If $B$ is regular, then $A$ is decidable.

*Problems 2–7 are on the following pages.*

2. ***More of the same.*** For each statement below, there are two boxes in the answer booklet labeled "Yes" and "No". Check "Yes" if the statement is ***always*** true and "No" otherwise, and give a ***brief*** (at most one short sentence) explanation of your answer. **Assume *P ≠ NP*.** If there is any other ambiguity or uncertainty about an answer, check "No".

   (a) Which of the following statements are true?

   - The solution to the recurrence $T(n) = 8T(n/2) + O(n^2)$ is $T(n) = O(n^2)$.
   - The solution to the recurrence $T(n) = 2T(n/8) + O(n^2)$ is $T(n) = O(n^2)$.
   - Every directed acyclic graph contains at least one sink.
   - Given *any* undirected graph $G$, we can compute a spanning tree of $G$ in $O(V + E)$ time using whatever-first search.
   - Suppose $A[1..n]$ is an array of integers. Consider the following recursive function:

   $$What(i,j) = \begin{cases} 0 & \text{if } i < 0 \text{ or } i > n \\ 0 & \text{if } j < 0 \text{ or } j > n \\ \max\left\{ \begin{array}{c} What(i, j-1) \\ What(i-1, j) \\ A[i] \cdot A[j] + What(i+1, j+1) \end{array} \right\} & \text{otherwise} \end{cases}$$

     We can memoize this function into an array $What[0..n, 0..n]$ in $O(n^2)$ time, by increasing $i$ in the outer loop and increasing $j$ in the inner loop.

   (b) Consider the following pair of languages:

   - DirHamPath $:= \{G \mid G \text{ is a directed graph with a Hamiltonian path}\}$
   - Acyclic $:= \{G \mid G \text{ is a directed acyclic graph}\}$

   (For concreteness, assume that in both of these languages, graphs are represented by their adjacency matrices.) Which of the following statements are true, assuming P ≠ NP?

   - Acyclic ∈ NP
   - Acyclic ∩ DirHamPath ∈ P
   - DirHamPath is decidable.
   - A polynomial-time reduction from DirHamPath to Acyclic would imply P=NP.
   - A polynomial-time reduction from Acyclic to DirHamPath would imply P=NP.

*Problems 3–7 are on the following pages.*

3. Describe and analyze an algorithm to determine whether the language accepted by a given DFA is finite or infinite. You can assume the input alphabet of the DFA is {0, 1}. *[Hint: DFAs are directed graphs.]*

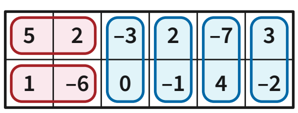
4. Suppose you are asked to tile a $2 \times n$ grid of squares with dominos ($1 \times 2$ rectangles). Each domino must cover exactly two grid squares, either horizontally or vertically, and each grid square must be covered by exactly one domino.

   Each grid square is worth some number of points, which could be positive, negative, or zero. The ***value*** of a domino tiling is the sum of the points in squares covered by vertical dominos, ***minus*** the sum of the points in squares covered by horizontal dominos.

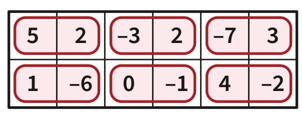
   Describe an algorithm to compute the largest possible value of a domino tiling of a given $2 \times n$ grid. Your input is an array $Points[1..2, 1..n]$ of point values.

   As an example, here are three domino tilings of the same $2 \times 6$ grid, along with their values. The third tiling is optimal; no other tiling of this grid has larger value. Thus, given this $2 \times 6$ grid as input, your algorithm should return the integer 16.

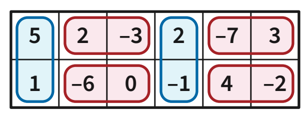
| | | | | | |
|---|---|---|---|---|---|
| 5 | 2 | −3 | 2 | −7 | 3 |
| 1 | −6 | 0 | −1 | 4 | −2 |



value = —6



value = 2



value = 16

5. ***Prove*** that the following problem (which we call MATCH) is NP-hard. The input is a finite set $S$ of strings, all of the same length $n$, over the alphabet {0, 1, 2}. The problem is to determine whether there is a string $w \in \{0, 1\}^n$ such that for every string $s \in S$, the strings $s$ and $w$ have the same symbol in at least one position.

   For example, given the set $S = \{$01220, 21110, 21120, 00211, 11101$\}$, the correct output is TRUE, because the string $w =$ 01001 matches the first three strings of $S$ in the second position, and matches the last two strings of $S$ in the last position. On the other hand, given the set $S = \{$2002, 2112, 2012, 2102$\}$, the correct output is FALSE.

   *[Hint: Describe a reduction from SAT (or 3SAT)]*

*Problems 6 and 7 are on the next page.*

6. Suppose you are given a height map of a mountain, in the form of an $n \times n$ grid of evenly spaced points, each labeled with an elevation value. You can safely hike directly from any point to any neighbor immediately north, south, east, or west, but only if the elevations of those two points differ by at most $\Delta$. (The value of $\Delta$ depends on your hiking experience and your physical condition.)

   Describe and analyze an algorithm to determine the longest hike from some point $s$ to some other point $t$, where the hike consists of an uphill climb (where elevations must increase at each step) followed by a downhill climb (where elevations must decrease at each step). Your input consists of an array $Elevation[1..n, 1..n]$ of elevation values, the starting point $s$, the target point $t$, and the parameter $\Delta$.

7. Recall that a ***run*** in a string $w \in \{0, 1\}^*$ is a maximal substring of $w$ whose characters are all equal. For example, the string `00011111110000` is the concatenation of three runs:

$$\texttt{00011111110000} = \texttt{000} \bullet \texttt{1111111} \bullet \texttt{0000}$$

   (a) Let $L_a$ denote the set of all strings in $\{0, 1\}^*$ where every `0` is followed immediately by at least one `1`.

   For example, $L_a$ contains the strings `010111` and `1111` and the empty string $\varepsilon$, but does not contain either `001100` or `1111110`.

   - Describe a DFA or NFA that accepts $L_a$ ***and***
   - Give a regular expression that describes $L_a$.

   (You do not need to prove that your answers are correct.)

   (b) Let $L_b$ denote the set of all strings in $\{0, 1\}^*$ whose run lengths are increasing; that is, every run except the last is followed immediately by a *longer* run.

   For example, $L_b$ contains the strings `0110001111` and `1100000` and `000` and the empty string $\varepsilon$, but does not contain either `000111` or `100011`.

   ***Prove*** that $L_b$ is not a regular language.