Polynomial Time Reductions

Lecture 24

Tuesday, November 19, 2024

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24.1
A quick review: Polynomials

What is a polynomial

A **polynomial** is a function of the form:

$$f(x) = \sum_{i=0}^{t} a_i x^i.$$

For our purposes, we can assume that $a_i \geq 0$, for all i.
A term $a_k x^t$ is a **monomial**.
The **degree** of $f(x)$ is t.
We have $f(n) = O(n^t)$.

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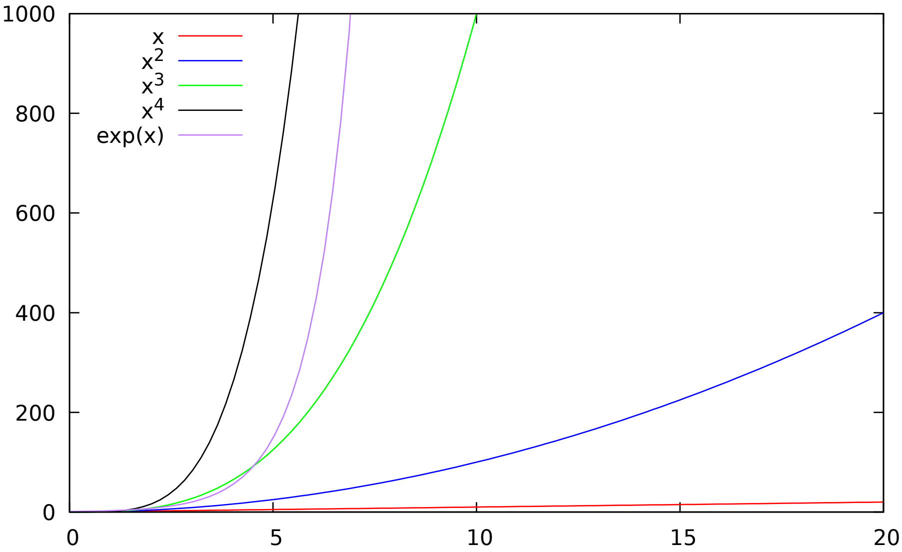
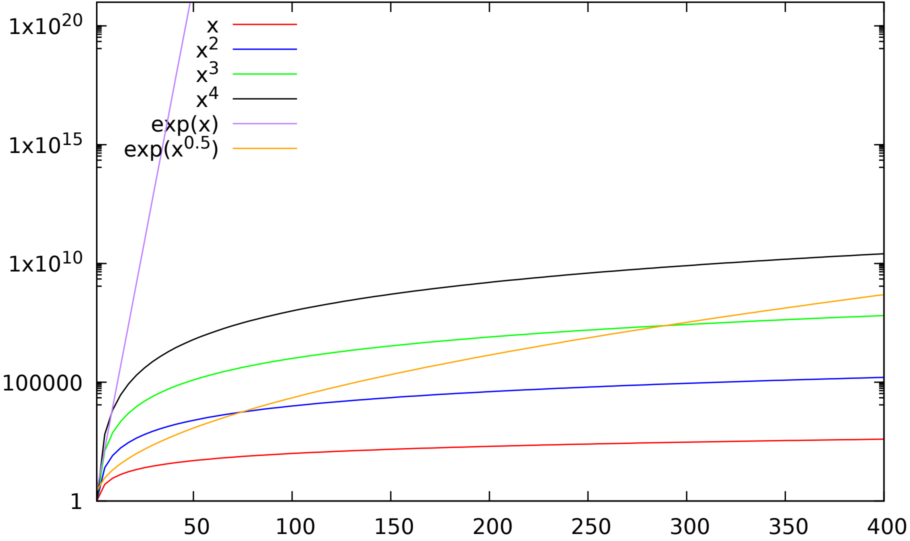


Polynomial time good, exponential time bad



Combining polynomials

Lemma 24.1.

If $f(x) = \sum_{i=0}^{d} \alpha_i x^i$ *is a polynomial of degree* d*, and* $g(y) = \sum_{i=0}^{d'} \beta_i y^i$ *is a polynomial of degree* d'*, then* $g(f(x))$ *is a polynomial of degree* $d'd$*.*

Proof.

Observe that $(f(x))^2 = \sum_{i=0}^{d} \sum_{j=0}^{d} \alpha_i \alpha_j x^{i+j}$ is a polynomial of degree $2d$, Arguing similarly, we have that $(f(x))^i$ is a polynomial of degree $i \cdot d$. Thus

$$g(f(x)) = \sum_{i=0}^{d'} \beta_i \bigl(f(x)\bigr)^i$$

is a sum of polynomials of degree $0, d, 2d, \ldots, d \cdot d'$, which is a polynomial of degree $d \cdot d'$ by collecting monomials of the same degree into a single monomial. □

24.2
(Polynomial Time) Reductions: Overview

Reductions

A reduction from Problem X to Problem Y means (informally) that if we have an algorithm for Problem Y, we can use it to find an algorithm for Problem X.

Using Reductions

1. We use reductions to find algorithms to solve problems.

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Using Reductions

1. We use reductions to find algorithms to solve problems.
2. We also use reductions to show that we can't find algorithms for some problems. (We say that these problems are hard.)

Reductions for decision problems/languages

For languages L_X, L_Y, a **reduction from L_X to L_Y** is:

1. An algorithm ...
2. Input: $w \in \Sigma^*$
3. Output: $w' \in \Sigma^*$
4. Such that:

$$\boxed{w \in L_X} \iff \boxed{w' \in L_Y}$$

(Actually, this is only one type of reduction, but this is the one we'll use most often.)

There are other kinds of reductions.

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Reductions for decision problems/languages

For decision problems $\boldsymbol{X}, \boldsymbol{Y}$, a **reduction from $\boldsymbol{X}$ to $\boldsymbol{Y}$** is:

1. An algorithm ...
2. Input: $\boldsymbol{I_X}$, an instance of $\boldsymbol{X}$.
3. Output: $\boldsymbol{I_Y}$ an instance of $\boldsymbol{Y}$.
4. Such that:

$$\boxed{\boldsymbol{I_Y} \text{ is YES instance of } \boldsymbol{Y}} \iff \boxed{\boldsymbol{I_X} \text{ is YES instance of } \boldsymbol{X}}$$

Using reductions to solve problems

1. $\mathcal{R}$: Reduction $X \to Y$
2. $\mathcal{A}_Y$: algorithm for Y:
3. $\implies$ New algorithm for X:

```
A_X(I_X):
        // I_X: instance of X.
        I_Y ⇐ R(I_X)
        return A_Y(I_Y)
```

If $\mathcal{R}$ and $\mathcal{A}_Y$ polynomial-time $\implies$ $\mathcal{A}_X$ polynomial-time.

Using reductions to solve problems

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A_X(I_X):
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        I_Y <= R(I_X)
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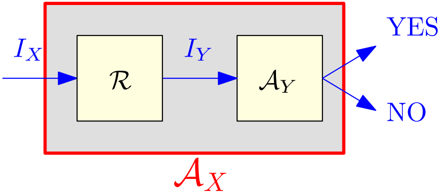

If $\mathcal{R}$ and $\mathcal{A}_Y$ polynomial-time $\implies$ $\mathcal{A}_X$ polynomial-time.

Comparing Problems

1. "Problem $\boldsymbol{X}$ is no harder to solve than Problem $\boldsymbol{Y}$".
2. If Problem $\boldsymbol{X}$ reduces to Problem $\boldsymbol{Y}$ (we write $\boldsymbol{X} \leq \boldsymbol{Y}$), then $\boldsymbol{X}$ cannot be harder to solve than $\boldsymbol{Y}$.
3. $\boldsymbol{X} \leq \boldsymbol{Y}$:
   3.1 $\boldsymbol{X}$ is no harder than $\boldsymbol{Y}$, or
   3.2 $\boldsymbol{Y}$ is at least as hard as $\boldsymbol{X}$.

24.3
Examples of Reductions

24.3.1
Independent Set and Clique

Independent Sets and Cliques

Given a graph G, a set of vertices V' is:

1. **independent set**: no two vertices of V' connected by an edge.

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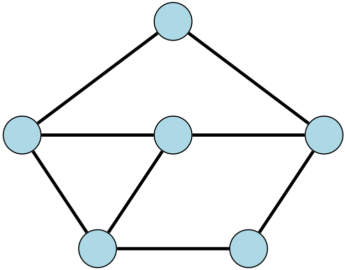

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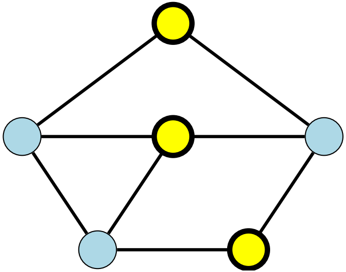

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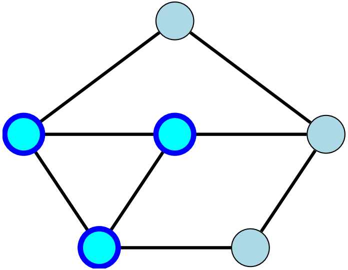

The **Independent Set** and **Clique** Problems

Problem: Independent Set

> **Instance:** A graph G and an integer k.
> **Question:** Does G has an independent set of size $\geq k$?

Problem: Clique

> **Instance:** A graph G and an integer k.
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Instance: A graph G and an integer k.
Question: Does G has an independent set of size $\geq k$?

Problem: Clique

Instance: A graph G and an integer k.
Question: Does G has a clique of size $\geq k$?

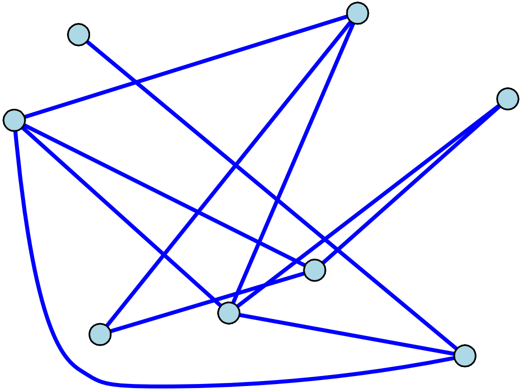
Recall

For decision problems $\boldsymbol{X, Y}$, a reduction from $\boldsymbol{X}$ to $\boldsymbol{Y}$ is:

1. An algorithm ...
2. that takes $\boldsymbol{I_X}$, an instance of $\boldsymbol{X}$ as input ...
3. and returns $\boldsymbol{I_Y}$, an instance of $\boldsymbol{Y}$ as output ...
4. such that the solution (YES/NO) to $\boldsymbol{I_Y}$ is the same as the solution to $\boldsymbol{I_X}$.

Reducing **Independent Set** to **Clique**

An instance of **Independent Set** is a graph G and an integer k.



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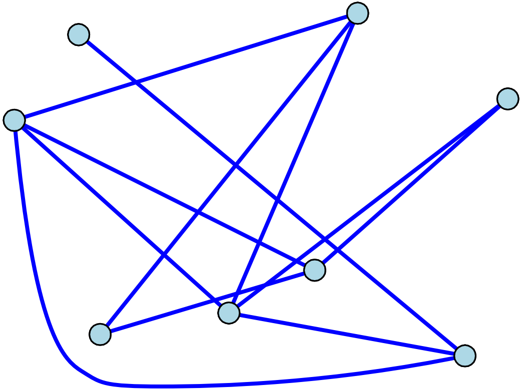

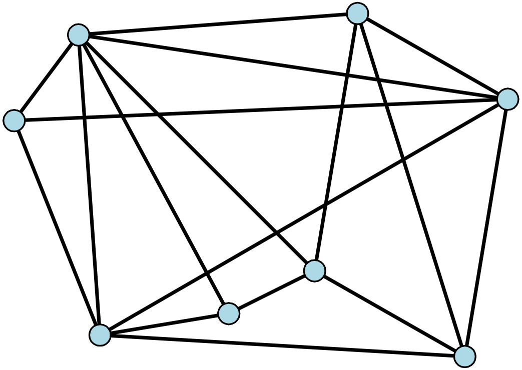

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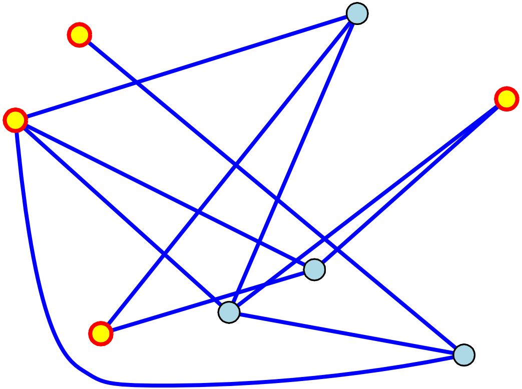

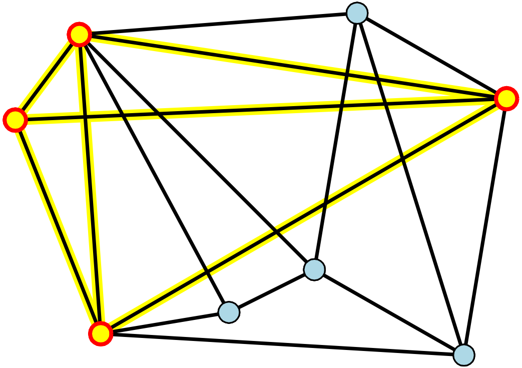

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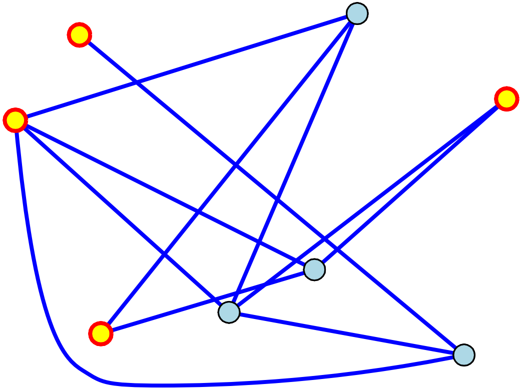

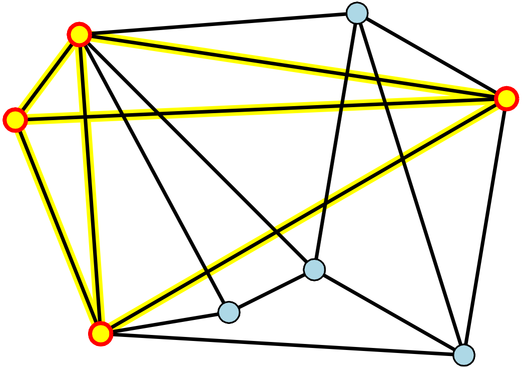

A independent set of size k in $G \iff$ A clique of size k in $\overline{G}$

Correctness of reduction

Lemma 24.1.

G has an independent set of size k $\iff$ $\overline{G}$ has a clique of size k.

Proof.

Need to prove two facts:
G has independent set of size at least k implies that $\overline{G}$ has a clique of size at least k.
$\overline{G}$ has a clique of size at least k implies that G has an independent set of size at least k.
Since $S \subseteq V$ is an independent set in G $\iff$ S is a clique in $\overline{G}$. □

Independent Set and Clique

1. **Independent Set** $\leq$ **Clique**.
   What does this mean?
2. If have an algorithm for **Clique**, then we have an algorithm for **Independent Set**.
3. **Clique** is at least as hard as **Independent Set**.
4. Also... **Clique** $\leq$ **Independent Set**. Why? Thus **Clique** and **Independent Set** are polnomial-time equivalent.

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Review: Independent Set and Clique

Assume you can solve the **Clique** problem in $T(n)$ time. Then you can solve the **Independent Set** problem in

(A) $O(T(n))$ time.

(B) $O(n \log n + T(n))$ time.

(C) $O(n^2 T(n^2))$ time.

(D) $O(n^4 T(n^4))$ time.

(E) $O(n^2 + T(n^2))$ time.

(F) Does not matter - all these are polynomial if $T(n)$ is polynomial, which is good enough for our purposes.

24.3.2
NFAs/DFAs and Universality

DFA Universality

A DFA M is universal if it accepts every string.
That is, $L(M) = \Sigma^*$, the set of all strings.

Problem 24.2 (DFA universality).

Input: *A DFA M.*
Goal: *Is M universal?*

How do we solve **DFA Universality**?
We check if M has any reachable non-final state.

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Input: *A NFA M.*
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How do we solve **NFA Universality**?
Reduce it to **DFA Universality**?
Given an NFA N, convert it to an equivalent DFA M, and use the **DFA Universality** Algorithm.
The reduction takes exponential time!
NFA Universality is known to be PSPACE-Complete and we do not expect a polynomial-time algorithm.

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24.4
Polynomial time reductions

24.4.1
A quick review of polynomial time reductions

Polynomial-time reductions

An algorithm is **efficient** if it runs in polynomial-time.

To find efficient algorithms for problems, we are only interested in polynomial-time reductions. Reductions that take longer are not useful.

If we have a polynomial-time reduction from problem X to problem Y (we write $X \leq_P Y$), and a poly-time algorithm $\mathcal{A}_Y$ for Y, we have a polynomial-time/efficient algorithm for X.

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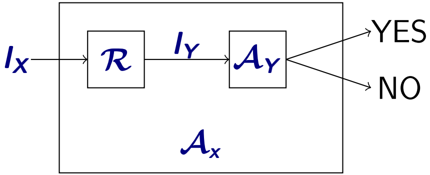

Polynomial-time Reduction

A polynomial time reduction from a decision problem X to a decision problem Y is an algorithm $\mathcal{A}$ that has the following properties:

1. given an instance I_X of X, $\mathcal{A}$ produces an instance I_Y of Y
2. $\mathcal{A}$ runs in time polynomial in $|I_X|$.
3. Answer to I_X YES $\iff$ answer to I_Y is YES.

Proposition 24.1.

If $X \leq_P Y$ then a polynomial time algorithm for Y implies a polynomial time algorithm for X.

Such a reduction is a **Karp reduction**. Most reductions we use are Karp reductions. Karp reductions are the same as mapping reductions when specialized to polynomial time for the reduction step.

Review question: Reductions again...

Let X and Y be two decision problems, such that X can be solved in polynomial time, and $X \leq_P Y$. Then

(A) Y can be solved in polynomial time.

(B) Y can NOT be solved in polynomial time.

(C) If Y is hard then X is also hard.

(D) None of the above.

(E) All of the above.

24.4.2
Polynomial-time reductions and hardness

Polynomial-time reductions and hardness

1. For decision problems $\boldsymbol{X}$ and $\boldsymbol{Y}$, if $\boldsymbol{X} \leq_P \boldsymbol{Y}$, and $\boldsymbol{Y}$ has an efficient algorithm, $\boldsymbol{X}$ has an efficient algorithm.
2. If you believe that **Independent Set** does NOT have an efficient algorithm...
3. Showed: **Independent Set** $\leq_P$ **Clique**
4. $\implies$ **Clique** should not be solvable in polynomial time.
5. If **Clique** had an efficient algorithm, so would **Independent Set**!

Proposition 24.2.

If $X \leq_P Y$ and X does not have an efficient algorithm, Y cannot have an efficient algorithm.

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Polynomial-time reductions and instance sizes

Proposition 24.3.

Let $\mathcal{R}$ be a polynomial-time reduction from X to Y. Then for any instance I_X of X, the size of the instance I_Y of Y produced from I_X by $\mathcal{R}$ is polynomial in the size of I_X.

Proof.

$\mathcal{R}$ is a polynomial-time algorithm and hence on input I_X of size $|I_X|$ it runs in time $p(|I_X|)$ for some polynomial $p()$.
I_Y is the output of $\mathcal{R}$ on input I_X.
$\mathcal{R}$ can write at most $p(|I_X|)$ bits and hence $|I_Y| \leq p(|I_X|)$. □

Note: Converse is not true. A reduction need not be polynomial-time even if output of reduction is of size polynomial in its input.

Polynomial-time reductions and instance sizes

Proposition 24.3.

Let $\mathcal{R}$ be a polynomial-time reduction from X to Y. Then for any instance I_X of X, the size of the instance I_Y of Y produced from I_X by $\mathcal{R}$ is polynomial in the size of I_X.

Proof.

$\mathcal{R}$ is a polynomial-time algorithm and hence on input I_X of size $|I_X|$ it runs in time $p(|I_X|)$ for some polynomial $p()$.
I_Y is the output of $\mathcal{R}$ on input I_X.
$\mathcal{R}$ can write at most $p(|I_X|)$ bits and hence $|I_Y| \leq p(|I_X|)$. □

Note: Converse is not true. A reduction need not be polynomial-time even if output of reduction is of size polynomial in its input.

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Polynomial-time Reduction

Definition 24.4.

A polynomial time reduction from a decision problem X to a decision problem Y is an algorithm $\mathcal{A}$ that has the following properties:

1. Given an instance I_X of X, $\mathcal{A}$ produces an instance I_Y of Y.
2. $\mathcal{A}$ runs in time polynomial in $|I_X|$. This implies that $|I_Y|$ (size of I_Y) is polynomial in $|I_X|$.
3. Answer to I_X YES $\iff$ answer to I_Y is YES.

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If $X \leq_P Y$ then a polynomial time algorithm for Y implies a polynomial time algorithm for X.

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Transitivity of Reductions

Proposition 24.6.

$X \leq_P Y$ and $Y \leq_P Z$ implies that $X \leq_P Z$.

Proof.

1. $\mathcal{R}_{X \to Y}$: Polynomial reduction that works in polynomial time $f(x)$.
2. $w \in L_X \iff w' = \mathcal{R}_{X \to Y}(w) \in L_Y$.
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8. Running time of $\mathcal{R}'(x)$ is $h(x) = g(f(x))$, which is a polynomial.

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Be careful about reduction direction

Note: $X \leq_P Y$ does not imply that $Y \leq_P X$ and hence it is very important to know the FROM and TO in a reduction.

To prove $X \leq_P Y$ you need to show a reduction FROM X TO Y
That is, show that an algorithm for Y implies an algorithm for X.

24.5
Independent Set and Vertex Cover

Vertex Cover

Given a graph $G = (V, E)$, a set of vertices S is:

1. A **vertex cover** if every $e \in E$ has at least one endpoint in S.

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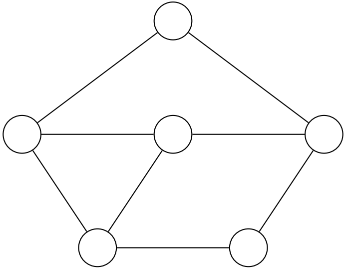

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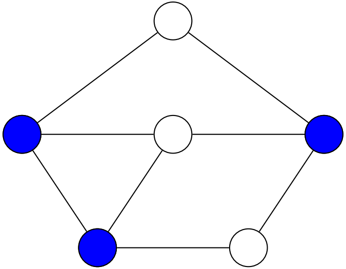

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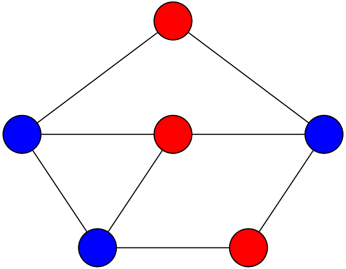

The **Vertex Cover** Problem

Problem 24.1 (Vertex Cover).

Input: *A graph G and integer k.*
Goal: *Is there a vertex cover of size $\leq k$ in G?*

Can we relate **Independent Set** and **Vertex Cover**?

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Can we relate **Independent Set** and **Vertex Cover**?

Relationship between...

Vertex Cover and Independent Set

Proposition 24.2.

Let $G = (V, E)$ be a graph. S is an Independent Set $\iff$ $V \setminus S$ is a vertex cover.

Proof.

($\Rightarrow$) Let S be an independent set

0.1 Consider any edge $uv \in E$.

0.2 Since S is an independent set, either $u \notin S$ or $v \notin S$.

0.3 Thus, either $u \in V \setminus S$ or $v \in V \setminus S$.

0.4 $V \setminus S$ is a vertex cover.

($\Leftarrow$) Let $V \setminus S$ be some vertex cover:

0.1 Consider $u, v \in S$

0.2 uv is not an edge of G, as otherwise $V \setminus S$ does not cover uv.

0.3 $\implies$ S is thus an independent set. □

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Independent Set $\leq_P$ Vertex Cover

1. G: graph with n vertices, and an integer k be an instance of the **Independent Set** problem.
2. G has an independent set of size $\geq k \iff G$ has a vertex cover of size $\leq n - k$
3. (G, k) is an instance of **Independent Set**, and $(G, n-k)$ is an instance of **Vertex Cover** with the same answer.
4. Therefore, **Independent Set** $\leq_P$ **Vertex Cover**. Also **Vertex Cover** $\leq_P$ **Independent Set**.

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Proving Correctness of Reductions

To prove that $X \leq_P Y$ you need to give an algorithm $\mathcal{A}$ that:

1. Transforms an instance I_X of X into an instance I_Y of Y.
2. Satisfies the property that answer to I_X is YES $\iff$ I_Y is YES.
   2.1 typical easy direction to prove: answer to I_Y is YES if answer to I_X is YES
   2.2 typical difficult direction to prove: answer to I_X is YES if answer to I_Y is YES (equivalently answer to I_X is NO if answer to I_Y is NO).
3. Runs in **polynomial** time.

24.6
The Satisfiability Problem (SAT)

24.6.1
CNF, SAT, 3CNF and 3SAT

Propositional Formulas

Definition 24.1.

Consider a set of boolean variables $x_1, x_2, \ldots x_n$.

1. A **literal** is either a boolean variable x_i or its negation $\neg x_i$.
2. A **clause** is a disjunction of literals.
   For example, $x_1 \vee x_2 \vee \neg x_4$ is a clause.
3. A **formula in conjunctive normal form** (CNF) is propositional formula which is a conjunction of clauses
   3.1 $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is a CNF formula.
4. A formula φ is a 3CNF:
   A CNF formula such that every clause has **exactly** 3 literals.
   4.1 $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_1)$ is a 3CNF formula, but
   $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is not.

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CNF is universal

Every boolean formula $f : \{0,1\}^n \to \{0,1\}$ can be written as a CNF formula.

x_1	x_2	x_3	x_4	x_5	x_6	$f(x_1, x_2, \ldots, x_6)$	
0	0	0	0	0	0	$f(0, \ldots, 0, 0)$	
0	0	0	0	0	1	$f(0, \ldots, 0, 1)$	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	
1	0	1	0	0	1	?	
1	0	1	0	1	0	0	
1	0	1	0	1	1	?	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	
1	1	1	1	1	1	$f(1, \ldots, 1)$	

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0	0	0	0	0	0	$f(0, \ldots, 0, 0)$	
0	0	0	0	0	1	$f(0, \ldots, 0, 1)$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
1	0	1	0	0	1	?	
1	0	1	0	1	0	0	
1	0	1	0	1	1	?	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
1	1	1	1	1	1	$f(1, \ldots, 1)$	

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x_1	x_2	x_3	x_4	x_5	x_6	$f(x_1, x_2, \ldots, x_6)$	$\overline{x_1} \vee x_2 \vee \overline{x_3} \vee x_4 \vee \overline{x_5} \vee x_6$
0	0	0	0	0	0	$f(0, \ldots, 0, 0)$	1
0	0	0	0	0	1	$f(0, \ldots, 0, 1)$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0	1	0	0	1	?	1
1	0	1	0	1	0	0	0
1	0	1	0	1	1	?	1
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CNF is universal

Every boolean formula $f : \{0,1\}^n \to \{0,1\}$ can be written as a CNF formula.

x_1	x_2	x_3	x_4	x_5	x_6	$f(x_1, x_2, \ldots, x_6)$	$\overline{x_1} \vee x_2 \vee \overline{x_3} \vee x_4 \vee \overline{x_5} \vee x_6$
0	0	0	0	0	0	$f(0, \ldots, 0, 0)$	1
0	0	0	0	0	1	$f(0, \ldots, 0, 1)$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0	1	0	0	1	?	1
1	0	1	0	1	0	0	0
1	0	1	0	1	1	?	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
1	1	1	1	1	1	$f(1, \ldots, 1)$	1

For every row that f is zero compute corresponding CNF clause.

CNF is universal

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x_1	x_2	x_3	x_4	x_5	x_6	$f(x_1, x_2, \ldots, x_6)$	$\overline{x_1} \vee x_2 \vee \overline{x_3} \vee x_4 \vee \overline{x_5} \vee x_6$
0	0	0	0	0	0	$f(0, \ldots, 0, 0)$	1
0	0	0	0	0	1	$f(0, \ldots, 0, 1)$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0	1	0	0	1	?	1
1	0	1	0	1	0	0	0
1	0	1	0	1	1	?	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
1	1	1	1	1	1	$f(1, \ldots, 1)$	1

For every row that f is zero compute corresponding CNF clause.
Take the and ($\bigwedge$) of all the CNF clauses computed

CNF is universal

Every boolean formula $f : \{0,1\}^n \rightarrow \{0,1\}$ can be written as a CNF formula.

x_1	x_2	x_3	x_4	x_5	x_6	$f(x_1, x_2, \ldots, x_6)$	$\overline{x_1} \vee x_2 \vee \overline{x_3} \vee x_4 \vee \overline{x_5} \vee x_6$
0	0	0	0	0	0	$f(0, \ldots, 0, 0)$	1
0	0	0	0	0	1	$f(0, \ldots, 0, 1)$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0	1	0	0	1	?	1
1	0	1	0	1	0	0	0
1	0	1	0	1	1	?	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
1	1	1	1	1	1	$f(1, \ldots, 1)$	1

For every row that f is zero compute corresponding CNF clause.
Take the and ($\bigwedge$) of all the CNF clauses computed

Resulting CNF formula equivalent to f.

Satisfiability

Problem: SAT

Instance: A CNF formula φ.
Question: Is there a truth assignment to the variable of φ such that φ evaluates to true?

Problem: 3SAT

Instance: A 3CNF formula φ.
Question: Is there a truth assignment to the variable of φ such that φ evaluates to true?

Satisfiability

SAT

Given a CNF formula φ, is there a truth assignment to variables such that φ evaluates to true?

Example 24.2.

1. $(x_1 \lor x_2 \lor \neg x_4) \land (x_2 \lor \neg x_3) \land x_5$ is satisfiable; take $x_1, x_2, \ldots x_5$ to be all true
2. $(x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2) \land (x_1 \lor x_2)$ is not satisfiable.

3SAT

Given a 3CNF formula φ, is there a truth assignment to variables such that φ evaluates to true?

(More on **2SAT** in a bit...)

Importance of **SAT** and **3SAT**

1. **SAT** and **3SAT** are basic constraint satisfaction problems.
2. Many different problems can reduced to them because of the simple yet powerful expressively of logical constraints.
3. Arise naturally in many applications involving hardware and software verification and correctness.
4. As we will see, it is a fundamental problem in theory of **NP-Complete**ness.

$z = \overline{x}$

Given two bits x, z which of the following **SAT** formulas is equivalent to the formula $z = \overline{x}$:

(A) $(\overline{z} \vee x) \wedge (z \vee \overline{x})$.

(B) $(z \vee x) \wedge (\overline{z} \vee \overline{x})$.

(C) $(\overline{z} \vee x) \wedge (\overline{z} \vee \overline{x}) \wedge (\overline{z} \vee \overline{x})$.

(D) $z \oplus x$.

(E) $(z \vee x) \wedge (\overline{z} \vee \overline{x}) \wedge (z \vee \overline{x}) \wedge (\overline{z} \vee x)$.

$z = x \wedge y$

Given three bits x, y, z which of the following **SAT** formulas is equivalent to the formula $z = x \wedge y$:

(A) $(\overline{z} \vee x \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(B) $(\overline{z} \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(C) $(\overline{z} \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(D) $(z \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(E) $(z \vee x \vee y) \wedge (z \vee x \vee \overline{y}) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y}) \wedge (\overline{z} \vee x \vee y) \wedge (\overline{z} \vee x \vee \overline{y}) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (\overline{z} \vee \overline{x} \vee \overline{y})$.

$\mathbf{z = x \vee y}$

Given three bits x, y, z which of the following **SAT** formulas is equivalent to the formula $z = x \vee y$:

(A) $(\overline{z} \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(B) $(\overline{z} \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(C) $(z \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(D) $(z \vee x \vee y) \wedge (z \vee x \vee \overline{y}) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y}) \wedge (\overline{z} \vee x \vee y) \wedge (\overline{z} \vee x \vee \overline{y}) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (\overline{z} \vee \overline{x} \vee \overline{y})$.

(E) $(\overline{z} \vee x \vee y) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee x \vee \overline{y}) \wedge (z \vee \overline{x} \vee \overline{y})$.

24.6.1.1
Review problems on CNF

$\mathbf{z = \overline{x}}$: Solution

Given two bits x, z which of the following **SAT** formulas is equivalent to the formula $z = \overline{x}$:

(A) $(\overline{z} \vee x) \wedge (z \vee \overline{x})$.

(B) $(z \vee x) \wedge (\overline{z} \vee \overline{x})$.

(C) $(\overline{z} \vee x) \wedge (\overline{z} \vee \overline{x}) \wedge (\overline{z} \vee \overline{x})$.

(D) $z \oplus x$.

(E) $(z \vee x) \wedge (\overline{z} \vee \overline{x}) \wedge (z \vee \overline{x}) \wedge (\overline{z} \vee x)$.

x	y	$z = \overline{x}$
0	0	0
0	1	1
1	0	1
1	1	0

$\mathbf{z = x \wedge y}$

Given three bits x, y, z which of the following **SAT** formulas is equivalent to the formula $z = x \wedge y$:

(A) $(\overline{z} \vee x \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(B) $(\overline{z} \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(C) $(\overline{z} \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(D) $(z \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(E) $(z \vee x \vee y) \wedge (z \vee x \vee \overline{y}) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y}) \wedge (\overline{z} \vee x \vee y) \wedge (\overline{z} \vee x \vee \overline{y}) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (\overline{z} \vee \overline{x} \vee \overline{y})$.

x	y	z	$z = x \wedge y$
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

$\mathbf{z = x \vee y}$

Given three bits x, y, z which of the following **SAT** formulas is equivalent to the formula $z = x \vee y$:

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(B) $(\overline{z} \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(C) $(z \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(D) $(z \vee x \vee y) \wedge (z \vee x \vee \overline{y}) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y}) \wedge (\overline{z} \vee x \vee y) \wedge (\overline{z} \vee x \vee \overline{y}) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (\overline{z} \vee \overline{x} \vee \overline{y})$.

(E) $(\overline{z} \vee x \vee y) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee x \vee \overline{y}) \wedge (z \vee \overline{x} \vee \overline{y})$.

x	y	z	$z = x \vee y$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

24.6.2
Reducing SAT to 3SAT

SAT $\leq_P$ 3SAT

How **SAT** is different from **3SAT**?

In **SAT** clauses might have arbitrary length: $\mathbf{1, 2, 3, \ldots}$ variables:

$$\left(x \vee y \vee z \vee w \vee u\right) \wedge \left(\neg x \vee \neg y \vee \neg z \vee w \vee u\right) \wedge \left(\neg x\right)$$

In **3SAT** every clause must have **exactly** **3** different literals.

To reduce from an instance of **SAT** to an instance of **3SAT**, we must make all clauses to have exactly **3** variables...

Basic idea

1. Pad short clauses so they have **3** literals.
2. Break long clauses into shorter clauses.
3. Repeat the above till we have a 3CNF.

SAT $\leq_P$ 3SAT

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Basic idea

1. Pad short clauses so they have **3** literals.
2. Break long clauses into shorter clauses.
3. Repeat the above till we have a $\mathrm{3CNF}$.

3SAT $\leq_P$ SAT

1. **3SAT** $\leq_P$ **SAT**.
2. Because...
   A **3SAT** instance is also an instance of **SAT**.

SAT $\leq_P$ 3SAT

Claim 24.3.

SAT $\leq_P$ 3SAT.

Given φ a SAT formula we create a 3SAT formula φ' such that

1. φ is satisfiable $\iff$ φ' is satisfiable.
2. φ' can be constructed from φ in time polynomial in $|\varphi|$.

Idea: if a clause of φ is not of length 3, replace it with several clauses of length exactly 3.

SAT $\leq_P$ 3SAT

Claim 24.3.

SAT $\leq_P$ **3SAT**.

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1. φ is satisfiable $\iff$ φ' is satisfiable.
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SAT $\leq_P$ 3SAT

Claim 24.3.

SAT $\leq_P$ **3SAT**.

Given φ a **SAT** formula we create a **3SAT** formula φ' such that

1. φ is satisfiable $\iff$ φ' is satisfiable.
2. φ' can be constructed from φ in time polynomial in $|\varphi|$.

Idea: if a clause of φ is not of length $\mathbf{3}$, replace it with several clauses of length exactly $\mathbf{3}$.

SAT $\leq_P$ 3SAT

A clause with two literals

Reduction Ideas: clause with 2 literals

1. Case clause with 2 literals: Let $c = \ell_1 \vee \ell_2$. Let u be a new variable. Consider

$$c' = \left(\ell_1 \vee \ell_2 \vee u\right) \wedge \left(\ell_1 \vee \ell_2 \vee \neg u\right).$$

2. Suppose $\varphi = \psi \wedge c$. Then $\varphi' = \psi \wedge c'$ is satisfiable $\iff$ φ is satisfiable.

SAT $\leq_P$ 3SAT

A clause with a single literal

Reduction Ideas: clause with 1 literal

1. Case clause with one literal: Let c be a clause with a single literal (i.e., $c = \ell$). Let u, v be new variables. Consider

$$c' = \left(\ell \vee u \vee v\right) \wedge \left(\ell \vee u \vee \neg v\right) \wedge \left(\ell \vee \neg u \vee v\right) \wedge \left(\ell \vee \neg u \vee \neg v\right).$$

2. Suppose $\varphi = \psi \wedge c$. Then $\varphi' = \psi \wedge c'$ is satisfiable $\iff$ φ is satisfiable.

SAT $\leq_P$ 3SAT

A clause with more than 3 literals

Reduction Ideas: clause with more than 3 literals

1. Case clause with five literals: Let $c = \ell_1 \vee \ell_2 \vee \ell_3 \vee \ell_4 \vee \ell_5$. Let u be a new variable. Consider

$$c' = \Big(\ell_1 \vee \ell_2 \vee \ell_3 \vee u\Big) \wedge \Big(\ell_4 \vee \ell_5 \vee \neg u\Big).$$

2. Suppose $\varphi = \psi \wedge c$. Then $\varphi' = \psi \wedge c'$ is satisfiable $\iff$ φ is satisfiable.

SAT $\leq_P$ 3SAT

A clause with more than 3 literals

Reduction Ideas: clause with more than 3 literals

1. Case clause with $k > 3$ literals: Let $c = \ell_1 \vee \ell_2 \vee \ldots \vee \ell_k$. Let u be a new variable. Consider

$$c' = \left(\ell_1 \vee \ell_2 \ldots \ell_{k-2} \vee u\right) \wedge \left(\ell_{k-1} \vee \ell_k \vee \neg u\right).$$

2. Suppose $\varphi = \psi \wedge c$. Then $\varphi' = \psi \wedge c'$ is satisfiable $\iff$ φ is satisfiable.

Breaking a clause

Lemma 24.4.

For any boolean formulas $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *and* $\boldsymbol{z}$ *a new boolean variable. Then*

$$\boldsymbol{X} \vee \boldsymbol{Y} \text{ is satisfiable}$$

if and only if, $\boldsymbol{z}$ *can be assigned a value such that*

$$\left(\boldsymbol{X} \vee z\right) \wedge \left(\boldsymbol{Y} \vee \neg z\right) \text{ is satisfiable}$$

(with the same assignment to the variables appearing in $\boldsymbol{X}$ *and* $\boldsymbol{Y}$*).*

SAT $\leq_P$ 3SAT (contd)

Clauses with more than 3 literals

Let $c = \ell_1 \vee \cdots \vee \ell_k$. Let $u_1, \ldots u_{k-3}$ be new variables. Consider

$$c' = \Big(\ell_1 \vee \ell_2 \vee u_1\Big) \wedge \Big(\ell_3 \vee \neg u_1 \vee u_2\Big) \wedge \Big(\ell_4 \vee \neg u_2 \vee u_3\Big) \wedge \cdots \wedge \Big(\ell_{k-2} \vee \neg u_{k-4} \vee u_{k-3}\Big) \wedge \Big(\ell_{k-1} \vee \ell_k \vee \neg u_{k-3}\Big).$$

Claim 24.5.

$\varphi = \psi \wedge c$ *is satisfiable* $\iff$ $\varphi' = \psi \wedge c'$ *is satisfiable.*

Another way to see it — reduce size of clause by one:

$$c' = \Big(\ell_1 \vee \ell_2 \ldots \vee \ell_{k-2} \vee u_{k-3}\Big) \wedge \Big(\ell_{k-1} \vee \ell_k \vee \neg u_{k-3}\Big).$$

An Example

Example 24.6.

$$\varphi = \Big(\neg x_1 \lor \neg x_4\Big) \land \Big(x_1 \lor \neg x_2 \lor \neg x_3\Big) \\ \land \Big(\neg x_2 \lor \neg x_3 \lor x_4 \lor x_1\Big) \land \Big(x_1\Big).$$

Equivalent form:

$$\psi = (\neg x_1 \lor \neg x_4 \lor z) \land (\neg x_1 \lor \neg x_4 \lor \neg z) \\ \land (x_1 \lor \neg x_2 \lor \neg x_3) \\ \land (\neg x_2 \lor \neg x_3 \lor y_1) \land (x_4 \lor x_1 \lor \neg y_1) \\ \land (x_1 \lor u \lor v) \land (x_1 \lor u \lor \neg v) \\ \land (x_1 \lor \neg u \lor v) \land (x_1 \lor \neg u \lor \neg v).$$

An Example

Example 24.6.

$$\varphi = \left(\neg x_1 \vee \neg x_4\right) \wedge \left(x_1 \vee \neg x_2 \vee \neg x_3\right) \wedge \left(\neg x_2 \vee \neg x_3 \vee x_4 \vee x_1\right) \wedge \left(x_1\right).$$

Equivalent form:

$$\begin{aligned}\psi = & (\neg x_1 \vee \neg x_4 \vee z) \wedge (\neg x_1 \vee \neg x_4 \vee \neg z) \\ & \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \\ & \wedge (\neg x_2 \vee \neg x_3 \vee y_1) \wedge (x_4 \vee x_1 \vee \neg y_1) \\ & \wedge (x_1 \vee u \vee v) \wedge (x_1 \vee u \vee \neg v) \\ & \wedge (x_1 \vee \neg u \vee v) \wedge (x_1 \vee \neg u \vee \neg v).\end{aligned}$$

An Example

Example 24.6.

$$\varphi = \Big(\neg x_1 \vee \neg x_4\Big) \wedge \Big(x_1 \vee \neg x_2 \vee \neg x_3\Big)$$
$$\wedge \Big(\neg x_2 \vee \neg x_3 \vee x_4 \vee x_1\Big) \wedge \Big(x_1\Big).$$

Equivalent form:

$$\psi = (\neg x_1 \vee \neg x_4 \vee z) \wedge (\neg x_1 \vee \neg x_4 \vee \neg z)$$
$$\wedge (x_1 \vee \neg x_2 \vee \neg x_3)$$
$$\wedge (\neg x_2 \vee \neg x_3 \vee y_1) \wedge (x_4 \vee x_1 \vee \neg y_1)$$
$$\wedge (x_1 \vee u \vee v) \wedge (x_1 \vee u \vee \neg v)$$
$$\wedge (x_1 \vee \neg u \vee v) \wedge (x_1 \vee \neg u \vee \neg v).$$

An Example

Example 24.6.

$$\varphi = \Big(\neg x_1 \vee \neg x_4\Big) \wedge \Big(x_1 \vee \neg x_2 \vee \neg x_3\Big) \\ \wedge \Big(\neg x_2 \vee \neg x_3 \vee x_4 \vee x_1\Big) \wedge \Big(x_1\Big).$$

Equivalent form:

$$\begin{aligned} \psi = & (\neg x_1 \vee \neg x_4 \vee z) \wedge (\neg x_1 \vee \neg x_4 \vee \neg z) \\ & \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \\ & \wedge (\neg x_2 \vee \neg x_3 \vee y_1) \wedge (x_4 \vee x_1 \vee \neg y_1) \\ & \wedge (x_1 \vee u \vee v) \wedge (x_1 \vee u \vee \neg v) \\ & \wedge (x_1 \vee \neg u \vee v) \wedge (x_1 \vee \neg u \vee \neg v). \end{aligned}$$

Overall Reduction Algorithm

Reduction from **SAT** to **3SAT**

```
ReduceSATTo3SAT(φ):
    // φ: CNF formula.
    for each clause c of φ do
        if c does not have exactly 3 literals then
            construct c' as before
        else
            c' = c
    ψ is conjunction of all c' constructed in loop
    return Solver3SAT(ψ)
```

Correctness (informal)

φ is satisfiable $\iff$ ψ is satisfiable because for each clause c, the new $3\mathrm{CNF}$ formula c' is logically equivalent to c.

24.6.3
2SAT

What about **2SAT**?

2SAT can be solved in polynomial time! (specifically, linear time!)

No known polynomial time reduction from **SAT** (or **3SAT**) to **2SAT**. If there was, then **SAT** and **3SAT** would be solvable in polynomial time.

Why the reduction from 3SAT to 2SAT fails?

Consider a clause $(x \vee y \vee z)$. We need to reduce it to a collection of $2\mathrm{CNF}$ clauses. Introduce a face variable α, and rewrite this as

$$(x \vee y \vee \alpha) \wedge (\neg\alpha \vee z) \qquad \text{(bad! clause with 3 vars)}$$

$$\text{or} \quad (x \vee \alpha) \wedge (\neg\alpha \vee y \vee z) \qquad \text{(bad! clause with 3 vars).}$$

(In animal farm language: **2SAT** good, **3SAT** bad.)

What about **2SAT**?

A challenging exercise: Given a **2SAT** formula show to compute its satisfying assignment...
(Hint: Create a graph with two vertices for each variable (for a variable x there would be two vertices with labels $x = \mathbf{0}$ and $x = \mathbf{1}$). For ever $\mathbf{2}\mathrm{CNF}$ clause add two directed edges in the graph. The edges are implication edges: They state that if you decide to assign a certain value to a variable, then you must assign a certain value to some other variable.
Now compute the strong connected components in this graph, and continue from there...)