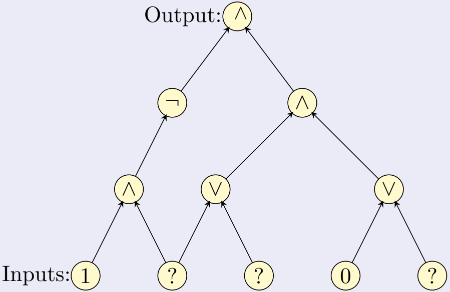
24.2
Circuit SAT

24.2.1
The circuit satisfiability (CSAT) problem

Circuits

Definition 24.1.

A circuit is a directed acyclic graph with



1. Input vertices (without incoming edges) labelled with **0**, **1** or a distinct variable.
2. Every other vertex is labelled $\vee$, $\wedge$ or $\neg$.
3. Single node output vertex with no outgoing edges.

Can safely assume every node has at most two incoming edges.

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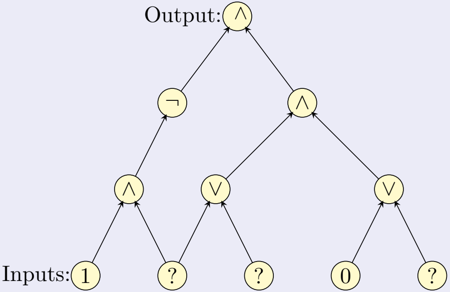

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CSAT: Circuit Satisfaction

Definition 24.2 (Circuit Satisfaction (CSAT).).

Given a circuit as input, is there an assignment to the input variables that causes the output to get value **1**?

Claim 24.3.

CSAT is in **NP**.

1. Certificate: Assignment to input variables.
2. Certifier: Evaluate the value of each gate in a topological sort of DAG and check the output gate value.

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Circuit SAT vs SAT

CNF formulas are a rather restricted form of Boolean formulas.

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Converting a CNF formula into a Circuit

3SAT $\leq_P$ CSAT

Given 3CNF formula φ with n variables and m clauses, create a Circuit C.

- Inputs to C are the n boolean variables $x_1, x_2, \ldots, x_n$
- Use NOT gate to generate literal $\neg x_i$ for each variable x_i
- For each clause $(\ell_1 \vee \ell_2 \vee \ell_3)$ use two OR gates to mimic formula
- Combine the outputs for the clauses using AND gates to obtain the final output

Example

3SAT $\leq_P$ CSAT

$$\varphi = \left(x_1 \vee \vee x_3 \vee x_4\right) \wedge \left(x_1 \vee \neg x_2 \vee \neg x_3\right) \wedge \left(\neg x_2 \vee \neg x_3 \vee x_4\right)$$

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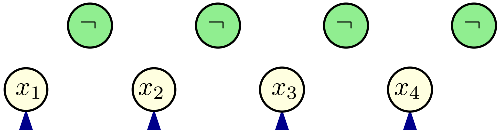
$$\varphi = \left(x_1 \lor \lor x_3 \lor x_4 \right) \land \left(x_1 \lor \neg x_2 \lor \neg x_3 \right) \land \left(\neg x_2 \lor \neg x_3 \lor x_4 \right)$$



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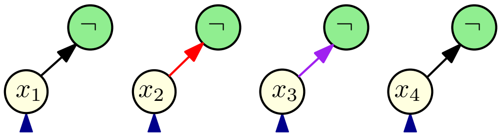

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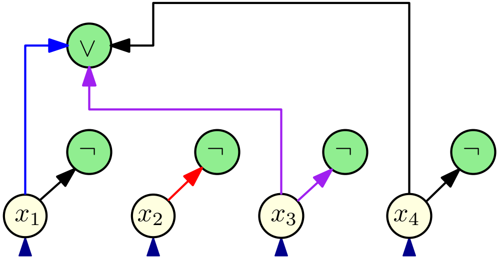

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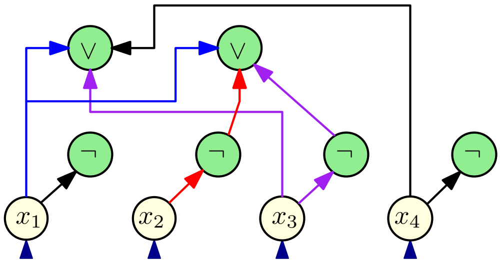
$$\varphi = \Big(x_1 \lor \lor x_3 \lor x_4\Big) \land \Big(x_1 \lor \neg x_2 \lor \neg x_3\Big) \land \Big(\neg x_2 \lor \neg x_3 \lor x_4\Big)$$



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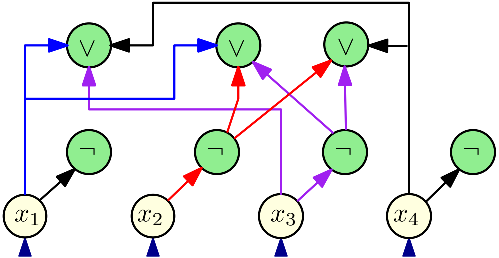

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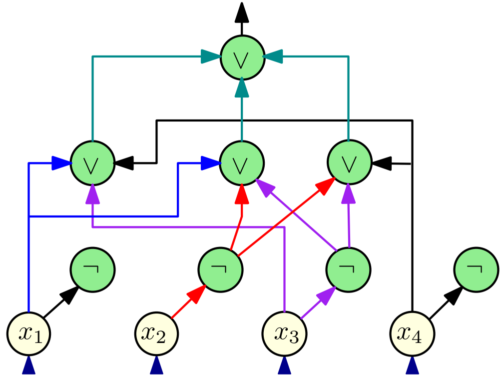

3SAT $\leq_P$ CSAT

Lemma 24.4.
SAT $\leq_P$ *3SAT* $\leq_P$ *CSAT*.

THE END

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(for now)