Algorithms & Models of Computation

CS/ECE 374, Fall 2020

Circuit satisfiability and Cook-Levin Theorem

Lecture 24

Thursday, December 3, 2020

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24.1
Recap

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NP: languages that have non-deterministic polynomial time algorithms

A language L is **NP-Complete** if and only if

- L is in **NP**
- for every L' in **NP**, $L' \leq_P L$

L is **NP-Hard** if for every L' in **NP**, $L' \leq_P L$.

Theorem 24.1 (Cook-Levin).
SAT is **NP-Complete***.*

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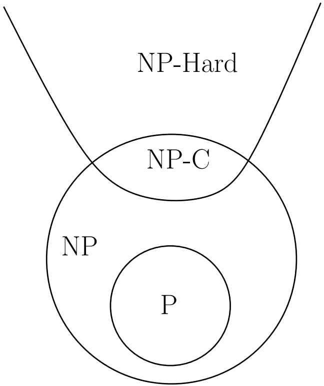
*SAT is **NP-Complete**.*

Pictorial View



P and **NP**

Possible scenarios:

1. **P** $=$ **NP**.
2. **P** $\neq$ **NP**

Question: Suppose **P** $\neq$ **NP**. Is every problem in **NP** $\setminus$ **P** also **NP-Complete**?

Theorem 24.2 (Ladner).

If **P** $\neq$ **NP** *then there is a problem/language* $X \in$ **NP** $\setminus$ **P** *such that* X *is not* **NP-Complete**.

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What do we know so far

1. **Independent Set** $\leq_P$ **Clique**, **Clique** $\leq_P$ **Independent Set**.
   $\Longrightarrow$ **Clique** $\approxeq_P$ **Independent Set**.
2. **Vertex Cover** $\leq_P$ **Independent Set**, **Independent Set** $\leq_P$ **Vertex Cover**.
   $\Longrightarrow$ **Independent Set** $\approxeq_P$ **Vertex Cover**.
3. **3SAT** $\leq_P$ **SAT**, **SAT** $\leq_P$ **3SAT** $\Longrightarrow$ **3SAT** $\approxeq_P$ **SAT**.
4. **3SAT** $\leq_P$ **Independent Set** .
   Exercise (or Cook-Levin theorem): **Independent Set** $\leq_P$ **SAT**
   $\Longrightarrow$ **3SAT** $\approxeq_P$ **Independent Set**.
5. **SAT** $\leq_P$ **Hamiltonian Cycle**
   Exercise (or Cook-Levin theorem): **Hamiltonian Cycle** $\leq_P$ **3SAT**
   $\Longrightarrow$ **Hamiltonian Cycle** $\approxeq_P$ **3SAT**
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NP Completeness

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SAT is **NPC**.

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THE END

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(for now)