Halting, Undecidability, and Maybe Some Complexity

Lecture 9

Tuesday, September 22, 2020

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Quote

"Young man, in mathematics you don't understand things. You just get used to them."
– John von Neumann.

9.1

Cantor's diagonalization argument

You can not count the real numbers

$I = (0, 1)$.
$\mathbb{N} = \{1, 2, 3, \ldots\}$ the integer numbers

Claim (Cantor)

$|\mathbb{N}| \neq |I|$

Claim (Warm-up)

$|\mathbb{N}| \leq |I|$

Proof.

$|\mathbb{N}| \leq |I|$ exists a one-to-one mapping from $\mathbb{N}$ to I. One such mapping is $f(i) = 1/i$, which readily implies the claim. $\square$

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You can not count the real numbers II

$\boldsymbol{I} = (0, 1)$, $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Claim (Cantor)

$|\mathbb{N}| \neq |\boldsymbol{I}|$, *where* $\boldsymbol{I} = (0, 1)$.

Proof.

Write every number in $(0, 1)$ in its decimal expansion. E.g., $1/3 = 0.33333333333333333333\ldots$.

Assume that $|\mathbb{N}| = |\boldsymbol{I}|$. Then there exists a one-to-one mapping $\boldsymbol{f} : \mathbb{N} \to \boldsymbol{I}$. Let β_i be the $\boldsymbol{i}$th digit of $\boldsymbol{f}(\boldsymbol{i}) \in (0, 1)$.

$d_i =$ any number in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \setminus \{d_{i-1}, \beta_i\}$

$D = 0.d_1 d_2 d_3 \ldots \in (0, 1)$.

D is a well defined unique number in $(0, 1)$,

But there is no j such that $f(j) = D$. A contradiction. $\square$

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$1/3 = 0.33333333333333333333\ldots$.
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The matrix...

	$f(1)$	$f(2)$	$f(3)$	$f(4)$	...
1	**1**	1	0	0	...
2	0	**1**	0	1	...
3	1	0	**1**	1	...
4	0	1	0	**0**	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The matrix...

	$f(1)$	$f(2)$	$f(3)$	$f(4)$	...
1	$\beta_1 = \mathbf{1}$	1	0	0	...
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3	1	0	$\beta_3 = \mathbf{1}$	1	...
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$D = 0.23232323\ldots$.
D can not be the i column, because $\beta_i \neq d_i$.

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$\implies \forall i \; \beta_i \neq d_i$.
$D = 0.23232323\ldots$.
D can not be the i column, because $\beta_i \neq d_i$.
But D can not be in the matrix...

The liar paradox

When one day an expedition was sent to the spatial coordinates that Voojagig had claimed for this planet they discovered only a small asteroid inhabited by a solitary old man who claimed repeatedly that nothing was true, though he was later discovered to be lying.
– The Hitchhiker Guide to the Galaxy

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2. Related to Russell's paradox.
3. Omnipotence paradox: Can [an omnipotent being] create a stone so heavy that it cannot lift it?

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THE END

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(for now)